Dear students,

on Friday March 15th I made a mistake in class when I said that $B_{\bar\rho}(\underline{x}, \varepsilon) = \prod_{\alpha \in J} B_{\bar d_\alpha}(x_\alpha, \varepsilon)$. This is FALSE.

Recall that $\bar\rho(\underline{x}, \underline{y}) := \sup_{\alpha \in J} (\bar d_\alpha(x_\alpha, y_\alpha))$.

Therefore we have:

$$\underline{y} \in B_{\bar\rho}(\underline{x}, \varepsilon) \Rightarrow \sup_{\alpha \in J}(\bar d_\alpha(x_\alpha, y_\alpha)) < \varepsilon$$

$$\Rightarrow \forall \alpha \in J,\ \bar d_\alpha(x_\alpha, y_\alpha) < \varepsilon$$

$$\Rightarrow \forall \alpha \in J,\ y_\alpha \in B_{\bar d_\alpha}(x_\alpha, \varepsilon)$$

Hence $\boxed{B_{\bar\rho}(\underline{x}, \varepsilon) \subset \prod_{\alpha \in J} B_{\bar d_\alpha}(x_\alpha, \varepsilon).}$ We only have an inclusion.

The reverse inclusion is not true, but we nevertheless have:

$$\boxed{\prod_{\alpha \in J} B_{\bar d_\alpha}(x_\alpha, \varepsilon/2) \subset B_{\bar\rho}(\underline{x}, \varepsilon)}$$

Namely: $\underline{y} \in \prod_{\alpha \in J} B_{\bar d_\alpha}(x_\alpha, \varepsilon/2) \Rightarrow \forall \alpha \in J,\ \bar d_\alpha(x_\alpha, y_\alpha) < \varepsilon/2$

$$\Rightarrow \sup_{\alpha \in J}(\bar d_\alpha(x_\alpha, y_\alpha)) \leq \varepsilon/2 < \varepsilon$$

$$\Rightarrow \underline{y} \in B_{\bar\rho}(\underline{x}, \varepsilon).$$

This proves the following (which I wrote in class without proving it):

The collections $\{B_{\bar\rho}(\underline{x}, \varepsilon) \mid \underline{x} \in \prod_{\alpha \in J} X_\alpha,\ \varepsilon > 0\}$ and $\{\prod_{\alpha \in J} B_{\bar d_\alpha}(x_\alpha, \varepsilon) \mid x_\alpha \in X_\alpha,\ \varepsilon > 0\}$ are both basis for the uniform topology on $\prod_{\alpha \in J} X_\alpha$.