# VI Introduction to algebraic topology

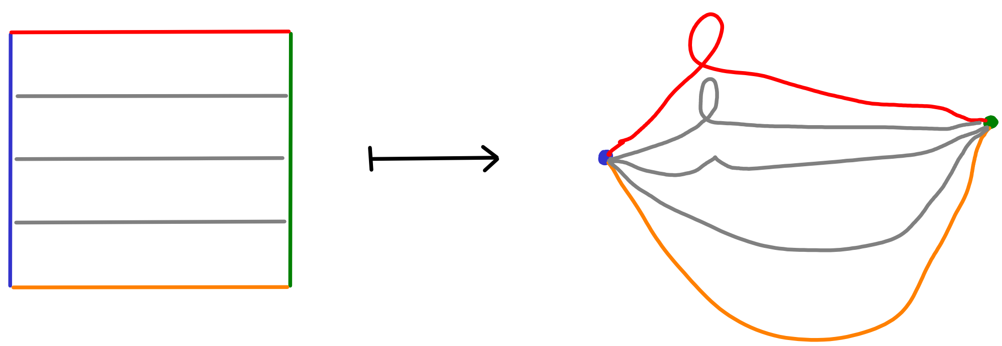
## 1] Path, homotopies and the fundamental group.

$X$ topological space. $x_0, x_1 \in X$. We denote by ${}_{x_0}X_{x_1}$ the set of paths $\gamma : [0,1] \to X$ from $x_0$ to $x_1$.

**Definition:** a (path-) homotopy from a path $\gamma_0 \in {}_{x_0}X_{x_1}$ to a path $\gamma_1 \in {}_{x_0}X_{x_1}$ is a continuous map $h : [0,1]^2 \to X$ such that:

(i) $h(t,0) = \gamma_0(t)$ and $h(t,1) = \gamma_1(t)$ $\quad (\forall t)$

(ii) $h(0,s) = x_0$ and $h(1,s) = x_1$ $\quad (\forall s)$



**Lemma:** The relation $\sim$ on ${}_{x_0}X_{x_1}$ defined by:

$\gamma \sim \gamma' \overset{\text{def}}{\iff}$ there exists a homotopy from $\gamma_0$ to $\gamma_1$

is an equivalence relation.

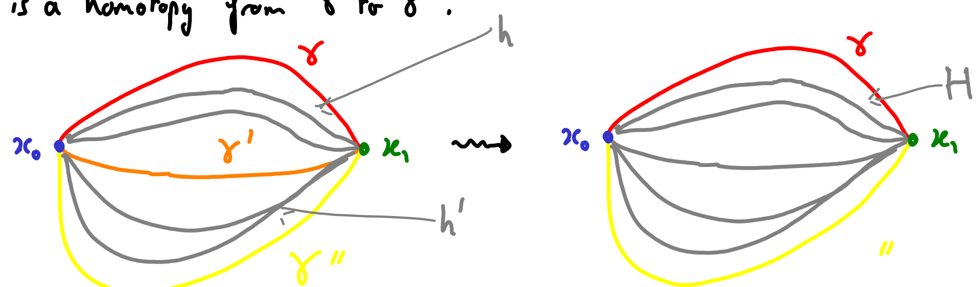
**Proof:**
- $\gamma_0 \sim \gamma_1 \iff \gamma_1 \sim \gamma_0$ (chang $h(t,s)$ to $h(t,1-s)$).
- $\gamma \sim \gamma$ $\quad (h(t,s) := \gamma(t))$.
- $\gamma \sim \gamma'$ & $\gamma' \sim \gamma'' \Rightarrow \gamma \sim \gamma''$.

Let $h$ be a homotopy from $\gamma$ to $\gamma'$ and $h'$ be a homotopy from $\gamma'$ to $\gamma''$.

Define $H(t,s) := \begin{cases} h(t,2s) & \text{if } 0 \leq s \leq \frac{1}{2} \\ h'(t,2s-1) & \text{if } \frac{1}{2} \leq s \leq 1 \end{cases}$

**Claim:** $H$ is a homotopy from $\gamma$ to $\gamma''$.

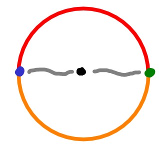
"proof":




□

actually true for any convex subset $A \subset \mathbb{R}^n$.

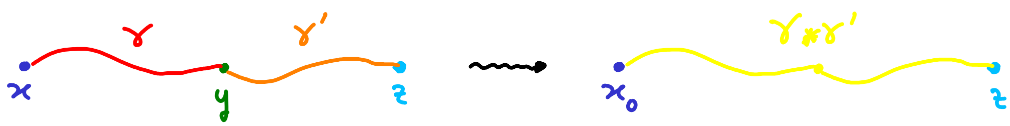
**Examples**:
- $X = \mathbb{R}^n$. Any two paths $\gamma, \gamma' \in {}_{x_0}X_{x_1}$ are homotopic:
  $$h(t,s) = \gamma(t) + s(\gamma'(t) - \gamma(t)).$$
- $X = \mathbb{R}^2 - \{(0,0)\}$. The paths $\gamma(t) = (\cos(\pi t), \sin(\pi t))$ and $\gamma'(t) = (\cos(\pi t), -\sin(\pi t))$ are not homotopic:



WE WILL PROVE THIS LATER.

We have seen a composition operation on homotopies between paths.
One can actually also compose the path themselves.

**Definition**: we fix $x, y, z \in X$, $\gamma \in {}_xX_y$ and $\gamma' \in {}_yX_z$.
We define $\gamma * \gamma' \in {}_xX_z$ as follows:
$$\gamma * \gamma'(t) := \begin{cases} \gamma(2t) & \text{if } 0 \leq t \leq \frac{1}{2} \\ \gamma'(2t-2) & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases}$$




**Claim**: the homotopy class of $\gamma * \gamma'$ does only depend on the homotopy classes of $\gamma$ and $\gamma'$.

**Proof**: Let $\gamma_0 \sim \gamma_1$ in ${}_xX_y$ and $\gamma_0' \sim \gamma_1'$ in ${}_yX_z$. We want to prove that $\gamma_0 * \gamma_0' \sim \gamma_1 * \gamma_1'$.
Let $h$ be a homotopy from $\gamma_0$ to $\gamma_1$ and $h'$ be a homotopy from $\gamma_0'$ to $\gamma_1'$.
Define
$$H(t,s) := \begin{cases} h(2t,s) & \text{if } 0 \leq t \leq \frac{1}{2}. \\ h'(2t-2,s) & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases}$$
**Claim**: $H$ is a homotopy from $\gamma_0 * \gamma_0'$ to $\gamma_1 * \gamma_1'$.
**Proof**: left as an exercise (encourage students to draw a picture). □

We have seen that one can inverse a homotopy from $\gamma$ to $\gamma'$ in order to get a homotopy from $\gamma'$ to $\gamma$.
One can actually also inverse paths:

For $\gamma \in {}_xX_y$ we define $\gamma^{-1} \in {}_yX_x$ as follows: $\gamma^{-1}(t) = \gamma(1-t)$.

**Claim**:
1) if $\gamma_0 \sim \gamma_1$ then $\gamma_0^{-1} \sim \gamma_1^{-1}$.
2) $(\gamma * \gamma')^{-1} = (\gamma')^{-1} * \gamma^{-1}$. [here we have an equality]
3) $\gamma * \gamma^{-1} \sim 1_x$, where $1_x(t) := x$ is the constant path at $x$.

Observe that $1_x$ is a neutral element for the composition:

For any $\gamma \in {}_xX_y$, $1_x * \gamma \sim \gamma$ and $\gamma * 1_y \sim \gamma$

Proof: We only prove that $1_x * \gamma \sim \gamma$.

Define $h(t,s) := \begin{cases} x & \text{if } 0 \leq t \leq \frac{s}{2} \\ \gamma\left(\frac{2t-s}{2-s}\right) & \text{if } \frac{s}{2} \leq t \leq 1 \end{cases}$.

Check that $h$ is a homotopy from $\gamma$ to $1_x * \gamma$. □

<u>Theorem</u>: composition $*$ is associative on homotopy classes of paths. Namely, if $x,y,z,w \in X$ and $\gamma \in {}_xX_y$, $\gamma' \in {}_yX_z$, $\gamma'' \in {}_zX_w$, then $(\gamma * \gamma') * \gamma'' \sim \gamma * (\gamma' * \gamma'')$.

<u>Proof</u>: $(\gamma * \gamma') * \gamma''(t) = \begin{cases} \gamma(4t) & \text{if } 0 \leq t \leq \frac{1}{4} \\ \gamma'(4t-1) & \text{if } \frac{1}{4} \leq t \leq \frac{1}{2} \\ \gamma''(2t-1) & \text{if } \frac{1}{2} \leq t \leq 1 \end{cases}$

$\gamma * (\gamma' * \gamma'')(t) = \begin{cases} \gamma(2t) & \text{if } 0 \leq t \leq \frac{1}{2} \\ \gamma'(4t-2) & \text{if } \frac{1}{2} \leq t \leq \frac{3}{4} \\ \gamma''(4t-3) & \text{if } \frac{3}{4} \leq t \leq 1 \end{cases}$

Let us introduce, for any $0 < u < v < 1$, the product

$\gamma \underset{u}{*} \gamma' \underset{v}{*} \gamma''(t) = \begin{cases} \gamma\left(\frac{t}{u}\right) & \text{if } 0 \leq t \leq u \\ \gamma'\left(\frac{t-u}{v-u}\right) & \text{if } u \leq t \leq v \\ \gamma''\left(\frac{t-v}{1-v}\right) & \text{if } v \leq t \leq 1 \end{cases}$

Observe that $(\gamma * \gamma') * \gamma'' = \gamma \underset{1/4}{*} \gamma' \underset{1/2}{*} \gamma''$ and $\gamma * (\gamma' * \gamma'') = \gamma \underset{1/2}{*} \gamma' \underset{3/4}{*} \gamma''$.

Let us fix $v$ and move $u \in (0,v)$:

Let $u : [0,1] \to [u_0, u_1]$ be any continuous map such that $u(0) = u_0$ and $u(1) = u_1$.

Then $(t,s) \mapsto \gamma \underset{u(s)}{*} \gamma' \underset{v}{*} \gamma''(t)$ defines a homotopy from $\gamma \underset{u_0}{*} \gamma' \underset{v}{*} \gamma''$ to $\gamma \underset{u_1}{*} \gamma' \underset{v}{*} \gamma''$.

Similarly one has a homotopy from $\gamma \underset{u}{*} \gamma' \underset{v_0}{*} \gamma''$ to $\gamma \underset{u}{*} \gamma' \underset{v_1}{*} \gamma''$ for $0 < u < v_0 \leq v_1 < 1$.

<u>Conclusion</u>: $(\gamma * \gamma') * \gamma''$ and $\gamma * (\gamma' * \gamma'')$ are both homotopic to $\gamma \underset{1/4}{*} \gamma' \underset{3/4}{*} \gamma''$. □

<u>Corollary</u>: for any $x \in X$, ${}_xX_x/\sim$ is a group, denoted $\pi_1(X,x)$ and called the <u>fundamental group</u> of $X$ at $x$.