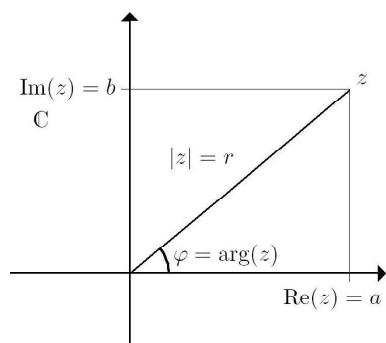


Lösungen zur Serie 1

Für die ersten 2 Aufgaben braucht man die folgenden Relationen:



Für $z = a + ib = \operatorname{Re}(z) + i \cdot \operatorname{Im}(z)$

$$z = re^{i\varphi} = |z|e^{i\varphi}$$

$$\varphi = \arg(z) \in [0, 2\pi)$$

$$\bar{z} = a - ib = re^{-i\varphi}$$

gelten folgende Beziehungen:

- $a^2 + b^2 = r^2$

- $\varphi = \begin{cases} \arctan(\frac{b}{a}), & \text{für } a > 0, b \text{ beliebig} \\ \arctan(\frac{b}{a}) + \pi, & \text{für } a < 0, b \geq 0 \\ \arctan(\frac{b}{a}) - \pi, & \text{für } a < 0, b < 0 \\ \frac{\pi}{2}, & \text{für } a = 0, b > 0 \\ -\frac{\pi}{2}, & \text{für } a = 0, b < 0 \\ \text{unbestimmt,} & \text{für } a = 0, b = 0 \end{cases}$

- $\cos(\varphi) = \frac{a}{r}$

- $\sin(\varphi) = \frac{b}{r}$

Bitte wenden!

1. a) $|\frac{1}{z}| = \frac{1}{|z|} = 3 \Rightarrow |z| = \frac{1}{3}$.

$$\arg(\bar{z}) = -\arg(z) = 240^\circ = -\varphi \quad \Rightarrow \varphi = 120^\circ = \frac{2}{3}\pi$$

Damit ist:

$$\operatorname{Re}(z) = \cos(\varphi) \cdot r = -\frac{1}{2} \cdot \frac{1}{3} = -\frac{1}{6}$$

$$\operatorname{Im}(z) = \sin(\varphi) \cdot r = \frac{\sqrt{3}}{2} \cdot \frac{1}{3} = \frac{\sqrt{3}}{6}$$

b) $z = \frac{(-1+i\sqrt{3}) \cdot (1-\sqrt{3}i)}{(1+\sqrt{3}i) \cdot (1-\sqrt{3}i)} = \frac{-1+2i\sqrt{3}+3}{4} = \frac{1}{2} + i\frac{\sqrt{3}}{2}$.

Damit ist:

$$\operatorname{Re}(z) = \frac{1}{2}$$

$$\operatorname{Im}(z) = \frac{\sqrt{3}}{2}$$

$$|z| = \sqrt{a^2 + b^2} = \sqrt{\frac{1}{4} + \frac{3}{4}} = 1$$

$$\arg(z) = \arctan(\sqrt{3}) + k \cdot \pi = \frac{\pi}{3}$$

c) $|i\bar{z}| = |i| \cdot |\bar{z}| = |\bar{z}| = |z| = 5$

$$\operatorname{Re}(i\bar{z}) = \operatorname{Re}(i(a - ib)) = \operatorname{Re}(b + ia) = b = \operatorname{Im}(z) = 3.$$

$\varphi = \arg(z) = \arcsin(\frac{b}{r}) = 36.87^\circ$ oder 143.13° (Man kann es nicht eindeutig entscheiden)

$$\operatorname{Re}(z) = r \cdot \cos(\varphi) = \pm 4$$

Siehe nächstes Blatt!

d) $\arg(i\bar{z}) = \arg(ire^{-i\varphi}) = \arg(e^{i\pi/2}re^{-i\varphi}) = \arg(e^{i(\pi/2-\varphi)}) = \pi/2 - \varphi$

Man benutzt $\arg(i\bar{z}) = 300^\circ = \frac{5\pi}{3}$:
 $\Rightarrow \varphi = \pi/2 - \frac{5}{3}\pi + k \cdot 2\pi = -\frac{7}{6}\pi + k \cdot 2\pi = \frac{5}{6}\pi = 150^\circ$ (k so wählen, dass $\varphi \in [0, 2\pi)$ ist)

$$2 = \operatorname{Im}\left(\frac{z}{i}\right) = \operatorname{Im}\left(-i\frac{z}{1}\right) = \operatorname{Im}(-i(a+ib)) = \operatorname{Im}(-ia+b) = -a = -\operatorname{Re}(z)$$

$$\Rightarrow \operatorname{Re}(z) = -2$$

$$\operatorname{Im}(z) = \operatorname{Re}(z) \cdot \tan(\varphi) = -2 \cdot \left(-\frac{1}{\sqrt{3}}\right) = \frac{2}{\sqrt{3}}$$

$$|z| = \sqrt{a^2 + b^2} = \sqrt{4 + \frac{4}{3}} = \frac{4}{\sqrt{3}}.$$

2. a) i)

$$\begin{aligned} z_1 &= \frac{-\sqrt{3} + 1 - i(\sqrt{3} + 1)}{1+i} \cdot \frac{1-i}{1-i} \\ &= \frac{-\sqrt{3} + 1 - i(\sqrt{3} + 1) - i(-\sqrt{3} + 1) - (\sqrt{3} + 1)}{2} \\ &= \frac{-2\sqrt{3} - 2i}{2} = -\sqrt{3} - i. \end{aligned}$$

ii)

$$z_2 = e^{-i\pi/4} = \frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}, \text{ da } |z| = 1 \text{ und } \varphi = \frac{7}{4}\pi.$$

iii)

$$z_3 = \frac{1}{e^{2\pi/3}} = e^{-2\pi/3} + 0 \cdot i \in \mathbb{R}$$

iv)

$$z_4 = ie^{-i\pi/6} = e^{i\pi/2} \cdot e^{-i\pi/6} = e^{i\pi/3} = \frac{1}{2} + i\frac{\sqrt{3}}{2}$$

Bitte wenden!

b) i)

$$\arg(z_1) = \varphi = \arctan\left(\frac{1}{\sqrt{3}}\right) + k \cdot \pi = 210^\circ = \frac{7\pi}{6} \text{ (Da } z_1 \text{ im 4. Quadranten liegt.)}$$

ii)

$$(z_2)^2 z_3 = \left(e^{-i\pi/4}\right)^2 \cdot e^{-2\pi/3} = e^{-2\pi/3} \cdot \underbrace{e^{-i\pi/2}}_{=-i} = -ie^{-2\pi/3}$$

iii)

$$\frac{z_3}{z_4} = e^{-2\pi/3} \cdot \frac{1}{e^{i\pi/3}} = e^{-2\pi/3} e^{-i\pi/3} = e^{-2\pi/3} \left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)$$

3. a) $\frac{1}{5+16i} = \frac{5-16i}{(5+16i)(5-16i)} = \frac{5-16i}{281} = \frac{5}{281} - i\frac{16}{281}$

$$\frac{1}{2+5i} = \frac{2-5i}{(2+5i)(2-5i)} = \frac{2-5i}{29} = \frac{2}{29} - i\frac{5}{29}$$

b) $(1+i)^n + (1-i)^n = (\sqrt{2}e^{i\pi/4})^n + (\sqrt{2}e^{-i\pi/4})^n = \sqrt{2}^n (e^{in\pi/4} + e^{-in\pi/4}) = 2^{n/2} \cdot 2 \cos(n\pi/4)$

c) $\frac{z+1}{z-1} = \frac{(z+1)(\bar{z}-1)}{|z-1|^2} = \frac{|z|^2 - z + \bar{z} - 1}{|z-1|^2} = \frac{|z|^2 - 2i\operatorname{Im}(z) - 1}{|z-1|^2} = \frac{a^2 + b^2 - 1}{(a-1)^2 + b^2} - i\frac{2b}{(a-1)^2 + b^2}$