Numerische Mathematik

Prüfung Frühjahr 2002

1)
$$\underbrace{\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} E \\ F \\ M \end{pmatrix}}_{x} = \underbrace{\begin{pmatrix} 42 \\ 45 \\ 18 \\ 40 \\ 22 \\ 58 \end{pmatrix}}_{b} \qquad A^TA = \begin{pmatrix} 3 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{pmatrix} \quad A^Tb = \begin{pmatrix} 145 \\ 158 \\ 185 \end{pmatrix}$$

$A^TA\,x = A^Tb$ (Gauss)

$$\left(\begin{array}{ccc|c} 3 & 2 & 2 & 145 \\ 2 & 4 & 2 & 158 \\ 2 & 2 & 4 & 185 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c} 3 & 2 & 2 & 145 \\ 0 & 8 & 2 & 184 \\ 0 & 2 & 8 & 205 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c} 3 & 2 & 2 & 145 \\ 0 & 4 & 1 & 92 \\ 0 & 0 & -15 & -318 \end{array}\right)$$

$M = 318/15 = 21.2 = 106/5$

$F = \frac{1}{4}(92 - M) = 17.7 = 177/10$

$E = \frac{1}{3}(145 - 2(M+F)) = 22.4 = 112/5$

b) $\kappa_2(A^TA) = \dfrac{\lambda_{max}}{\lambda_{min}}$

$$\det\begin{pmatrix} 3-\lambda & 2 & 2 \\ 2 & 4-\lambda & 2 \\ 2 & 2 & 4-\lambda \end{pmatrix} = 0$$

$(3-\lambda)((4-\lambda)^2 - 2\cdot 2) - 2(2(4-\lambda) - 2\cdot 2) + 2(2\cdot 2 - (4-\lambda)\cdot 2) = 0$

$(3-\lambda)((4-\lambda)^2 - 4) - 4(2-\lambda - (-2+\lambda)) = 0$

$(3-\lambda)(4-\lambda)(4-\lambda) - 4(3-\lambda) - 4(2(2-\lambda)) = 0$

$(3-\lambda)(16 - 8\lambda + \lambda^2 - 4) - 8(2-\lambda) = 0$

$(3-\lambda)(6-\lambda)(2-\lambda) - 8(2-\lambda) = 0 \quad = (2-\lambda)[\lambda^2 - 9\lambda + 10] = 0$

$\lambda_1 = 2$ ist a solution

$\lambda_{2,3} = \dfrac{9 \pm \sqrt{81-40}}{2} = 7.7016,\ 1.2984$

$\kappa(A^TA) = \dfrac{\lambda_2}{\lambda_3} = 5.9314$

a) $A = D + L + R$

$x^{k+1} = -D^{-1}(L+R)\,x^{k} + D^{-1}b \qquad D^{-1} = \frac{1}{2} I \leftarrow I = \text{identity matrix}$

$$T = -\frac{1}{2}(L+R) = \frac{1}{2}\begin{pmatrix} 0 & 1 & & \\ 1 & 0 & 1 & \\ & & \ddots & \\ & & & 1 \\ & & 1 & 0 \end{pmatrix}, \quad c = D^{-1}b = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{pmatrix}$$

bi) $n = 2$

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

$$T = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Rightarrow \lambda^2 - \frac{1}{4} = 0 \qquad \rho(T) = \frac{1}{2} < 1 \quad \therefore \text{converges}$$

$$x^{k+1} = -(D+L)^{-1} R\, x^{k} + (D+L)^{-1} b$$

$$D+L = \begin{pmatrix} 2 & 0 \\ -1 & 2 \end{pmatrix} \qquad (D+L)^{-1} = \begin{pmatrix} 1/2 & 0 \\ 1/4 & 1/2 \end{pmatrix} \qquad T = -\begin{pmatrix} 1/2 & 0 \\ 1/4 & 1/2 \end{pmatrix}\begin{pmatrix} 0 & 1/2 \\ 0 & 1/4 \end{pmatrix}$$

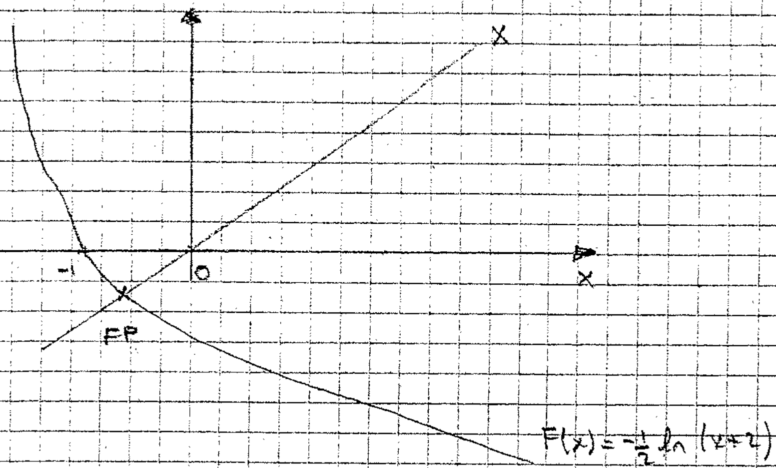
$$= \begin{pmatrix} 0 & -1/4 \\ 0 & -1/4 \end{pmatrix}$$

$$\rho(T) = \frac{1}{4} < 1 \quad \therefore \text{converges}$$

bii) $x^{1} = \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3/2 \\ 1 \end{pmatrix}$ Jacobi

$$x^{1} = \begin{pmatrix} 0 & 1/2 \\ 0 & 1/4 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 1/2 & 0 \\ 1/4 & 1/2 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 + 1/2 \\ 1/2 + 3/4 \end{pmatrix} = \begin{pmatrix} 3/2 \\ 5/4 \end{pmatrix}$$

3a)



b) $F(x)$ differentiable for $x > 2$

We need contraction k

$$k : |F'(x)| = \frac{1}{2(x+2)} < 1 \implies x > -\frac{3}{2}$$

$F'(x) < 0 \implies F(x)$ monoton calculate right boundary value

$$\left.\begin{array}{ll} F(-3/2) = -1/2 \ln(1/2) = 0.3465 & > -1.5 \\ F(10) = -\frac{1}{2}\ln(12) = -1.2442 & < 10 \end{array}\right\} \implies x = -3/2$$

c) $f'(x) = \frac{1}{x+2} + 2$ $x^1 = x^0 - \frac{f(x)}{f'(x)} = 9 - \frac{\ln(11) + 2}{1/11 + 2} = -0.7555$

i) $x_1 = F(x_0) = -\frac{1}{2}\ln(11) = -1.1989$

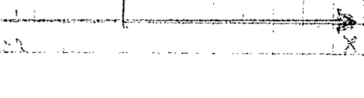
$f(x) = \ln(x+2)$, $I := [-2, 10]$

$f'(x) = \frac{1}{x+2} + 2$

$f' > 0$ in I

$$f'(x) = 1 \iff \frac{1}{x+2} + 2 = 1 \iff \frac{1}{x+2} = -1$$

$$\iff 1 = -x - 2 \iff x = -3 \notin I$$



$|f'(x)| > 1$ for $x \in]-2, \infty[$

f has no contraction

$I_x :=]x, 10]$ $f(10) = \ln(12) + 20 > 10$, here exists no x with $f(I_x) \subset I_x$

4a) $p(x_i) = e^{-x_i^2}$

a) Approximate with interpolation formula

$$p(0) = a = 1$$
$$p(1/2) = a + 1/4 b + 1/16 c = e^{-1/4} = 0.7788$$
$$p(1) = a + b + c = e^{-1} = 0.3679$$

$$\begin{pmatrix} 1/4 & 1/16 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} b \\ c \end{pmatrix} = \begin{pmatrix} e^{-1/4} - 1 \\ e^{-1} - 1 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} 1/4 & 1/16 & e^{-1/4} - 1 \\ 0 & 3/4 & e^{-1} - 1 - 4e^{-1/4} + 4 \end{array}\right)$$

$$\therefore \quad c = 4/3\,(e^{-1} - 4e^{-1/4} + 4) = 0.3369$$

$$b = 4\,(e^{-1/4} - 1 - c/16) = -0.9690$$

$$I = \int_0^1 p(x)\,dx = \left[ax + \frac{bx^3}{3} + \frac{cx^5}{5}\right] = a + \frac{b}{3} + \frac{c}{5} = 0.74437$$

b) $$I = \int_0^1 e^{-x^2}\,dx = \frac{1}{2}\int_{-1}^{+1} e^{-x^2}\,dx = \frac{1}{2}\sum_{i=1}^{5} w_i e^{-x_i^2} = 0.7468$$

a) For students who attempt part a) using Lagrange interpolation it is important that they interpolate $u = x^2$

$$p(u) = \sum_i f_i \prod_{j \neq i} \frac{u - u_j}{u_i - u_j}$$

This is because we require a polynomial of the form $p(x) = a_0 + bx^2 + cx^4$, standard Lagrange would give $a + bx + cx^2$.

$$p(u) = e^{-0}\,\frac{u - 1/4}{-1/4}\,\frac{u-1}{-1} + e^{-\frac{1}{4}}\,\frac{u-0}{1/4 - 0}\,\frac{u-1}{1/4 - 1} + e^{-1}\,\frac{u-0}{1-0}\,\frac{u - 1/4}{1 - 1/4}$$

$$= 0.3369u^2 - 0.9690u + 1 \quad \text{(as part a))}$$

5 a)

$$\begin{array}{c|cc} c_1 & a_{11} & a_{12} \\ c_2 & a_{21} & a_{22} \\ \hline & b_1 & b_2 \end{array} \qquad \begin{array}{c|cc} 0 & 0 & 0 \\ 2/3 & 1/3 & 1/3 \\ \hline & 1/4 & 3/4 \end{array}$$

$$K_1 = f(t_n + c_1 h,\ x_n + h(a_{11}K_1 + a_{12}K_2)) = f(t_n, x_n) = \frac{x_n^2}{t_n}$$

$$K_2 = f(t_n + c_2 h,\ x_n + h(a_{21}K_1 + a_{22}K_2)) = f\left(t_n + \frac{2h}{3},\ x_n + \frac{h}{3}(K_1 + K_2)\right)$$

$$= \frac{\left[x_n + \frac{h}{3}\left(\frac{x_n^2}{t_n} + K_2\right)\right]^2}{t_n + \frac{2h}{3}}$$

$$x_{n+1} = x_n + h(b_1 K_1 + b_2 K_2) = x_n + \frac{h}{4}(K_1 + 3K_2)$$

$$x_1 = x_0 + \frac{h}{4}\left(\frac{x_0^2}{t_0} + 3\frac{\left(x_0 + \frac{h}{3}\left(\frac{x_0^2}{t_0} + K_2\right)\right)^2}{t_0 + \frac{2h}{3}}\right)$$

b) Model problem $\dot{x} = \lambda x$

$$\Rightarrow K_1 = f(x) = \lambda x_n$$

$$K_2 = f\left(x + \frac{h}{3}(\lambda x_n + K_2)\right) = \lambda\left(x + \frac{h}{3}(\lambda x_n + K_2)\right) = \lambda x_n\left(1 + \frac{\lambda h}{3}\right) + \frac{\lambda h}{3}K_2$$

$$K_2 = \lambda x_n \frac{(3 + \lambda h)}{3 - \lambda h}$$

$$x_{n+1} = x_n + \frac{h}{4}\left(\lambda x_n + 3\lambda x_n \frac{3 + \lambda h}{3 - \lambda h}\right) = x_n \underbrace{\left(1 + \frac{\lambda h}{4} + \frac{3\lambda h}{4}\,\frac{3 + \lambda h}{3 - \lambda h}\right)}_{R}$$

let $\mu = \lambda h$

$$R(\mu) = 1 + \frac{\mu}{4} + \frac{3\mu}{4}\,\frac{3 + \mu}{3 - \mu} = 1 + \frac{\mu}{4}\left(\frac{(3 - \mu) + 3(3 + \mu)}{3 - \mu}\right) = 1 + \frac{\mu}{4}\,\frac{9 + 2\mu}{3 - \mu}$$

$$= \frac{1}{2}\left(\frac{6 - 2\mu + 6\mu + \mu^2}{3 - \mu}\right).$$

c) $\mu \in \mathbb{R}$ find ∂S where $S := \{\mu \mid |R(\mu)| < 1\}$ so $\partial S := \{\mu \mid |R| < 1\}$

$$\mu^2 + 4\mu + 6 = 6 - 2\mu \qquad \mu^2 + 6\mu = 0 \qquad \mu = \begin{cases} -6 \\ 0 \end{cases}$$

6a) $u_{xx} = -1$

$\Rightarrow u_x = -x + a$

$u = -\frac{x^2}{2} + ax + b$

We know that $u(\frac{1}{2}) = -\frac{1/4}{2} + \frac{1}{2}a + b = 1$ $\quad u(-1/2) = -\frac{1/4}{2} - \frac{1}{2}a + b = 1$

$2 = 2b - 2\frac{1/4}{2}$

$2 + \frac{1}{4} = 2b \Rightarrow b = \frac{9}{8}$ and $1 + \frac{1}{8} - \frac{9}{8} = \frac{1}{2}a \quad a = 0$

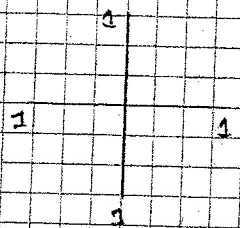
$\therefore u = -\frac{x^2}{2} + \frac{9}{8}$

$u(0) = 9/8$

ii) $u''(0) = \frac{u(-0.5) - 2u(0) + u(0.5)}{\Delta x^2} = \frac{1 - 2\cdot 9/8 + 1}{1/4} = -1$

$u(0) = 9/8$

d)



$-(u_{xx} + u_{yy}) = -\Delta u|_{(0,0)} \approx -\frac{u(-0.5,0) + u(0.5,0) + u(0,-0.5) + u(0,0.5) - 4u(0,0)}{(1/2)^2}$

$= -\frac{4(1 - u(0,0))}{1/4} = -16(1 - u(0,0)) = 1$

$u(0,0) = 17/16$