Frühjahr 2003

① SOR : $A = D + L + R$

$$x^{n+1} = (D+\omega L)^{-1}\left(-\omega R + (1-\omega) D\right) x^n + (D+\omega L)^{-1} \omega b$$

a) $\omega = 1 \Rightarrow$ GS

$$x^{n+1} = \overbrace{-(D+L)^{-1} R}^{T} x^n + \overbrace{(D+L)^{-1} b}^{c}$$

$$D = \begin{pmatrix} 3 & 0 \\ 0 & 1/3 \end{pmatrix}, \quad L = \begin{pmatrix} 0 & 0 \\ 1/9 & 0 \end{pmatrix} \Rightarrow D+L = \begin{pmatrix} 3 & 0 \\ 1/9 & 1/3 \end{pmatrix}$$

$$(D+L)^{-1} = \begin{pmatrix} 1/3 & 0 \\ -1/9 & 3 \end{pmatrix}$$

$$T = -(D+L)^{-1} R = \begin{pmatrix} 1/3 & 0 \\ -1/9 & 3 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1/3 \\ 0 & 1/9 \end{pmatrix}$$

$$c = (D+L)^{-1} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2/3 \\ 25/9 \end{pmatrix}$$

$$x^1 = \begin{pmatrix} 0 & -1/3 \\ 0 & 1/9 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 2/3 \\ 25/9 \end{pmatrix} = \frac{1}{9} \begin{pmatrix} 3 \\ 26 \end{pmatrix}$$

b) $D = \begin{pmatrix} 3 & 0 \\ 0 & 1/3 \end{pmatrix}, \quad L = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad R = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

$$T = D^{-1}\left(-\omega R + (1-\omega) D\right)$$

$$= \begin{pmatrix} 1/3 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 3(1-\omega) & -\omega \\ 0 & \frac{1-\omega}{3} \end{pmatrix} = \begin{pmatrix} 1-\omega & -\omega/3 \\ 0 & 1-\omega \end{pmatrix}$$

$\lambda_{1,2} = 1-\omega, \quad \rho(T) = |1-\omega|$, smallest with $\omega = 1$

②

a) $\|r\| = \min \iff \|\underbrace{U^T r}_{\tilde{r}}\| = \min \quad (U \in O(4))$

$r = Ax - c = USV^T x - c$

$\tilde{r} = S\underbrace{V^T x}_{y} - \underbrace{U^T c}_{d}$

$= Sy - d$

$$d = U^T c = \frac{1}{5}\begin{pmatrix} 1 & -2 & -4 & +2 \\ +2 & 1 & +2 & 4 \\ -4 & -2 & 1 & +2 \\ -2 & 4 & -2 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} -3 \\ 4 \\ -3 \\ -4 \end{pmatrix}$$

$\Rightarrow 5y_1 + 3/5 = 0 \qquad y_1 = -3/25 = 0.12$

$2y_2 - 4/5 = 0 \qquad y_2 = 2/5 = 0.4$

$y_3 = \kappa, \quad \kappa \in \mathbb{R}$

$$y = \frac{1}{25}\begin{pmatrix} -3 \\ 10 \\ \kappa \end{pmatrix} = \begin{pmatrix} -3/25 \\ 2/5 \\ \kappa \end{pmatrix}$$

$y = V^T x \quad x = Vy \quad (V \in O(3))$

$$x = \frac{1}{3}\begin{pmatrix} 2 & -2 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 2 \end{pmatrix}\begin{pmatrix} -3/25 \\ 2/5 \\ \kappa \end{pmatrix} = \frac{1}{75}\begin{pmatrix} -26 \\ 4 \\ 17 \end{pmatrix} + \frac{1}{3}\begin{pmatrix} \kappa \\ -2\kappa \\ 2\kappa \end{pmatrix}$$

$\|r\| = \min \iff \|\tilde{r}\| = \min \Rightarrow y_3 = 0$

$$\Rightarrow x = \frac{1}{75}\begin{pmatrix} -26 \\ 4 \\ 17 \end{pmatrix} = \begin{pmatrix} -0.3467 \\ 0.0533 \\ 0.2267 \end{pmatrix}$$

(3)

i	1	2	3	4
x	0	1	2	3
f	0	3	-1	0

$h_i = 1$

$a_i = 4$

$b_i = 1$

$c_1 = \frac{3-0}{1} = 3$ $\qquad d_2 = 3(c_1 + c_2) = -3$

$c_2 = \frac{-1-3}{1} = -4$ $\qquad d_3 = 3(c_2 + c_3) = -9$

$c_3 = \frac{0-(-1)}{1} = 1$

interior equations $\begin{pmatrix} 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \end{pmatrix} \begin{pmatrix} f_1' \\ f_2' \\ f_3' \\ f_4' \end{pmatrix} = \begin{pmatrix} -3 \\ -9 \end{pmatrix}$

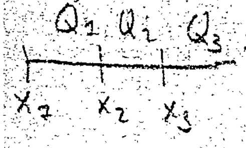
a) $d_1 = 2c_1 + \frac{1}{2}(c_1 + c_2) = 6 + \frac{-1}{2} = \frac{11}{2}$

$d_4 = 2c_3 + \frac{1}{2}(c_3 + c_2) = 2 + \frac{-3}{2} = \frac{1}{2}$

$A = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 2 & 1 \end{pmatrix}, \quad d = \begin{pmatrix} 11/2 \\ -3 \\ -9 \\ 1/2 \end{pmatrix}$

b) $Q_i(t) = f_i(1 - 3t^2 + 2t^3) + f_{i+1}(3t^2 - 2t^3)$ $\qquad h_i = 1 \ \forall i$

$+ h_i g_i (t - 2t^2 + t^3) + h_i g_{i+1}(-t^2 + t^3)$ $\qquad \Rightarrow \ddot{Q}_1(0) \doteq \ddot{Q}_3(1)$



$\dot{Q}_i(t) = f_i(-6t + 6t^2) + f_{i+1}(6t - 6t^2)$

$+ h_i g_i (1 - 4t + 3t^2) + h_i g_{i+1}(-2t + 3t^2)$

$\ddot{Q}_i(t) = f_i(-6 + 12t) + f_{i+1}(6 - 12t)$

$+ h_i g_i (-4 + 6t) + h_i g_{i+1}(-2 + 6t)$

$\ddot{Q}_1(0) = -6f_1 + 6f_2 - 4g_1 - 2g_2$, $\quad \ddot{Q}_3(1) = 6f_3 - 6f_4 + 2g_3 + 4g_4$ (But $g_4 = g_1$)

$\Rightarrow 4g_1 + g_2 + g_3 = 12$ $\qquad A = \begin{pmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 4 \end{pmatrix} \quad b = \begin{pmatrix} 12 \\ -3 \\ -9 \end{pmatrix}$

4 a) $f' \approx \dfrac{\dfrac{\sin(0.2)\cos(1.2)}{0.2} - \dfrac{\sin(-0.2)\cos(0.8)}{-0.2}}{0.4} = -0.83031138945$

b) $i=1 \quad h = \frac{3}{2} h_0 = 1.5 \times 0.2 = 0.3$

$$f' \approx \dfrac{\dfrac{\sin(0.3)\cos(1.3)}{0.3} - \dfrac{\sin(-0.3)\cos(0.7)}{-0.3}}{0.6} = -0.8165277845183$$

$i=2 \quad h = \left(\frac{3}{2}\right)^2 h_0 = \frac{9}{4} \times 0.2 = 0.45$

$$f' \approx \dfrac{\dfrac{\sin(0.45)\cos(1.45)}{0.45} - \dfrac{\sin(-0.45)\cos(0.55)}{-0.45}}{0.9} = -0.78618324047 5$$

c) $R_{00} = -0.8303\ldots$
$R_{10} = -0.8165\ldots$
$R_{20} = -0.78618\ldots$

$R_{00}, R_{10} \to R_{11}$; $R_{10}, R_{20} \to R_{21}$; $R_{11}, R_{21} \to R_{22}$

$a = 3/2$
$a^2 = 9/4$
$1/a^2 = 4/9$

$$R_{11} = R_{10} + \frac{1}{(4/9)^1 - 1}(R_{10} - R_{00}) = R_{10} - \frac{9}{5} R_{10} + \frac{9}{5} R_{00}$$

$$= \frac{9}{5} R_{00} - \frac{4}{5} R_{10} = -0.84133759395 4$$

$$R_{21} = R_{20} + \frac{1}{(4/9)^1 - 1}(R_{20} - R_{10})$$

$$= \frac{9}{5} R_{10} - \frac{4}{5} R_{20} = -0.840803488952$$

$$R_{22} = R_{21} - \frac{1}{(4/9)^2 - 1}(R_{21} - R_{11}) = \frac{81}{65} R_{11} - \frac{16}{65} R_{21} = 0.841469068955$$

exact $= -0.841470 9$

(5) $\dot{x} = f(t, x)$ $\begin{array}{c|c} 1/2 & 1/2 \\ \hline & 1 \end{array}$

a) $k_1 = f(t_n + h/2, x_n + h/2\, k_1)$

$x_{n+1} = x_n + h k_1 = x_n + h f(t_n + h/2, x_n + h/2\, k_1)$

b) $k_1 = \frac{x_{n+1} - x_n}{h}$

$x_{n+1} = x_n + h f\left(t_n + \frac{h}{2}, x_n + \frac{h}{2}\left(\frac{x_{n+1} - x_n}{h}\right)\right)$

$= x_n + h f\left(t_n + \frac{h}{2}, \frac{x_n + x_{n+1}}{2}\right)$

c) implicit

k_1 depends on k_1

x_{n+1} depends on x_{n+1}

on the diagonal $a_{11} = 1/2 \neq 0$

x_{n+1} arises in $f(\cdot, \cdot)$

d) $x(h) \approx x_1 = x_0 + h\left(-2\left(t_0 + \frac{h}{2}\right)\left(\frac{x_0 + x_1}{2}\right)^2\right)$ $\quad t_0 = 0$, $x_0 = 1$

$= 1 + h\left(-2\frac{h}{2}\left(\frac{1 + x_1}{2}\right)^2\right)$

$x_1 = 1 - \frac{h^2}{4}(1 + x_1)^2$ Fixpoint iteration for x_1

FPI $x_1^{(0)} = x_0 = 1$

$x_1^{(1)} = 1 - \frac{h^2}{4}\left(1 + x_1^{(0)}\right)^2 = 1 - h^2$

$x_1^{(2)} = 1 - \frac{h^2}{4}\left(1 + x_1^{(1)}\right)^2 = 1 - \frac{h^2}{4}\left(1 + (1 - h^2)\right)^2 = 1 - \frac{h^2}{4}(2 - h^2)^2$

$= 1 - \frac{h^2}{4}(4 - 4h^2 + h^4) = 1 - h^2 + h^4 - \frac{h^6}{4}$

(6)

$$-\Delta u(x,y) \approx \frac{1}{h^2}\Big(4u(x,y) - u(x+h,y) - u(x-h,y) - u(x,y+h) - u(x,y-h)\Big)$$

a) $h = 1/2$ $\quad 1/h^2 = 4 \quad \Delta u = 0$ in D, $\quad u = 1$ on ∂D

(1) $4(4u_1 - u_2 - 3) = 0$

(2) $4(4u_2 - u_1 - u_3 - 2) = 0 \quad \Rightarrow \quad A = \begin{pmatrix} 4 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 4 \end{pmatrix}, \quad b = \begin{pmatrix} 3 \\ 2 \\ 3 \end{pmatrix}$

(3) $4(4u_3 - u_2 - 3) = 0$

b) $f_{(1)} = 1/2 + 1/2 = 1 \quad f_2 = 3/2 \quad f_3 = 2$

(1) $4(4u_1 - u_2) = 1$

(2) $4(4u_2 - u_1 - u_3) = 3/2 \quad A = \begin{pmatrix} 4 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 4 \end{pmatrix} \quad b = \begin{pmatrix} 1/4 \\ 3/8 \\ 1/2 \end{pmatrix}$

(3) $4(4u_3 - u_2) = 2$