Numerische Mathematik

Frühjahr 2004

① $x_1 + 4x_2^2 - 3 = 0$

$4x_2 + x_1 + 2 = 0$

$$f(x) = \begin{pmatrix} x_1 + 4x_2^2 - 3 \\ 4x_2 + x_1 + 2 \end{pmatrix}$$

$$x^{k+1} = x^k + \Delta^k$$

where

$$J(x^k)\,\Delta^k = -f(x^k)$$

and

$$J = \begin{pmatrix} \partial f_1/\partial x_1 & \partial f_1/\partial x_2 \\ \partial f_2/\partial x_1 & \partial f_2/\partial x_2 \end{pmatrix} = \begin{pmatrix} 1 & 8x_2 \\ 1 & 4 \end{pmatrix}$$

iteration 1

$$x^0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad J(x^0) = \begin{pmatrix} 1 & 0 \\ 1 & 4 \end{pmatrix} \quad f(x^0) = \begin{pmatrix} -3 \\ 2 \end{pmatrix}$$

$$J(x^0)\,\Delta^0 = -f(x^0)$$

$$\begin{pmatrix} 1 & 0 \\ 1 & 4 \end{pmatrix} \Delta^0 = \begin{pmatrix} 3 \\ -2 \end{pmatrix} \Rightarrow \Delta^0 = \frac{1}{4}\begin{pmatrix} 4 & 0 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ -2 \end{pmatrix} = \begin{pmatrix} 3 \\ -\frac{5}{4} \end{pmatrix}, \quad x^1 = \begin{pmatrix} 3 \\ -5/4 \end{pmatrix}$$

iteration 2

$$J(x^1) = \begin{pmatrix} 1 & -10 \\ 1 & 4 \end{pmatrix} \quad f(x^1) = \begin{pmatrix} 6.25 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -10 \\ 1 & 4 \end{pmatrix} \Delta^1 = \begin{pmatrix} -6.25 \\ 0 \end{pmatrix} \qquad \Delta^1 = \frac{1}{14}\begin{pmatrix} 4 & 10 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} -6.25 \\ 0 \end{pmatrix} \approx \begin{pmatrix} -1.7857 \\ 0.446 \end{pmatrix}$$

$$x^1 = \begin{pmatrix} 1.2143 \\ -0.80357 \end{pmatrix}$$

b) The iteration fails when choosing a start value $x_2 = 1/2$. In this case $\det J = 0$ and the inverse of J [illegible]

② $A = \begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix} \quad b = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$

$D = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix} \quad L = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad R = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

Jacobi

$T = -D^{-1}(L+R)$

$= -\begin{pmatrix} 1/2 & 0 \\ 0 & 1/4 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1/2 \\ -1/4 & 0 \end{pmatrix}$

$\text{eig}(T) \Rightarrow \lambda^2 - 1/8 = 0$

$\lambda = \pm \frac{1}{\sqrt{8}} \approx \pm 0.3535 \quad \max|\lambda_i| = 0.3535 \leq 1$

converges

Gauss-Siedal

$T = -(D+L)^{-1}R$

$= -\begin{pmatrix} 2 & 0 \\ 1 & 4 \end{pmatrix}^{-1}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = -\frac{1}{8}\begin{pmatrix} 4 & 0 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1/2 \\ 0 & 1/8 \end{pmatrix}$

$\text{eig}(T) \Rightarrow \lambda\left(\frac{1}{8} - \lambda\right) = 0$

$\lambda = 0, \frac{1}{8} \quad \max|\lambda_i| = \frac{1}{8} \leq 1$ ∴ converges

b) Both schemes converge.

Jacobi

$Dx^{k+1} = -(L+R)x^k + b$

$x^{k+1} = Tx^k + D^{-1}b$

$x^1 = \begin{pmatrix} 0 & -1/2 \\ -1/4 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1/2 & 0 \\ 0 & 1/4 \end{pmatrix}\begin{pmatrix} 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1/4 \end{pmatrix}$

Gauss-Siedal

$x^{k+1} = Tx^k + (D+L)^{-1}b$

$x^1 = \begin{pmatrix} 0 & -1/2 \\ \ldots \end{pmatrix}\begin{pmatrix} 1 \\ \ldots \end{pmatrix} + \frac{1}{8}\begin{pmatrix} 4 & 0 \\ \ldots \end{pmatrix}\begin{pmatrix} 2 \\ \ldots \end{pmatrix} = \begin{pmatrix} 1/2 \\ \ldots \end{pmatrix}$

Convergence

$$\|e^k\|_2 = \|T^k e^0\|_2 \qquad \|e^k\|_2 \leq \|T\|_2^k \|e^0\|_2$$

$$\frac{\|e^k\|_2}{\|e^0\|_2} \leq \|T\|_2^k \qquad k \geq -\frac{6 \log 10}{\log \|T\|_2}$$

$$\|T\|_2 \doteq \sqrt{\max(\operatorname{eig}(T^tT))}$$

Jacobi

$$T^tT = \begin{pmatrix} 0 & -7/4 \\ -7/2 & 0 \end{pmatrix}\begin{pmatrix} 0 & -7/2 \\ -7/4 & 0 \end{pmatrix} = \begin{pmatrix} 7/16 & 0 \\ 0 & 7/4 \end{pmatrix}$$

$\operatorname{eig}(T^tT) \Rightarrow$

$$\left(\frac{7}{16} - \lambda\right)\left(\frac{7}{4} - \lambda\right) = 0$$

$$\lambda = \frac{7}{16}, \frac{7}{4}$$

$\|T\|_2 = 7/2 \quad k \geq 19.93 \approx 20$

Gauss - Siedal

$$T^tT = \begin{pmatrix} 0 & 0 \\ -7/2 & 7/8 \end{pmatrix}\begin{pmatrix} 0 & -7/2 \\ 0 & 7/8 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0.265625 \end{pmatrix}$$

$\operatorname{eig}(T^tT) \Rightarrow$

$$\lambda(0.265625 - \lambda) = 0$$

$$\lambda = \frac{17}{64}, 0$$

$\|T\|_2 = 0.5153 \qquad k \geq 20.84 \approx 21$

③

i	1	2	3	4	5	$(n=5)$
x_i	0	1/4	1/2	3/4	1	
f_i	-1	-1/2	0	1/2	1	

From the lecture notes we have the n-2 equations

$$b_1 f_1' + a_1 f_2' + b_2 f_3' = d_2$$
$$b_2 f_2' + a_2 f_3' + b_3 f_4' = d_3$$
$$b_3 f_3' + a_3 f_4' + b_4 f_5' = d_4$$

In matrix notation

$$\begin{pmatrix} b_1 & a_1 & b_2 & 0 & 0 \\ 0 & b_2 & a_2 & b_3 & 0 \\ 0 & 0 & b_3 & a_4 & b_4 \end{pmatrix} \begin{pmatrix} f_1' \\ f_2' \\ f_3' \\ f_4' \\ f_5' \end{pmatrix} = \begin{pmatrix} d_2 \\ d_3 \\ d_4 \end{pmatrix}$$

But we also know that $f_1' = f_5' = 0$. Hence we have 3 equations and 3 unknowns

$$\begin{pmatrix} a_1 & b_2 & 0 \\ b_2 & a_2 & b_3 \\ 0 & b_3 & a_3 \end{pmatrix} \begin{pmatrix} f_2' \\ f_3' \\ f_4' \end{pmatrix} = \begin{pmatrix} d_2 \\ d_3 \\ d_4 \end{pmatrix}$$

$$a_1 = a_2 = a_3 = \frac{2}{h} + \frac{2}{4} = \frac{4}{h} = 16$$

$$b_1 = b_2 = b_3 = \frac{1}{h} = 4$$

$$c_1 = \frac{f_2 - f_1}{h^2} = \frac{16}{2} = 8, \quad c_2 = \frac{f_3 - f_2}{h^2} = 16\left(0 + \frac{1}{2}\right) = 8, \quad c_3 = \frac{f_4 - f_3}{h^2} = 16\left(\frac{1}{2} - 0\right) = 8$$

$$c_4 = \frac{f_5 - f_4}{h^2} = 16\left(1 - \frac{1}{2}\right) = 8$$

$$d_2 = 3(c_1 + c_2) = 48, \quad d_3 = 3(c_2 + c_3) = 48 \quad d_4 = 3(c_3 + c_4) = 48$$

$$\begin{pmatrix} 16 & 4 & 0 \\ 4 & 16 & 4 \\ 0 & 4 & 16 \end{pmatrix} \begin{pmatrix} f_2' \\ f_3' \\ f_4' \end{pmatrix} = \begin{pmatrix} 48 \\ 48 \\ 48 \end{pmatrix} \rightarrow \begin{pmatrix} f_2' \\ f_3' \\ f_4' \end{pmatrix} = \begin{pmatrix} 2.5714 \\ 1.7143 \\ 2.5714 \end{pmatrix}$$

Evaluate at $x = 0.1$

First interval $x_i \leq x \leq x_{i+1}$

$0 \leq x \leq 0.25$

$t = \frac{0.1 - 0}{0.25} = 0.4$

$a_0 = -1$ $\quad b_{-1} = 0$

$b_0 = 1/2$ $\quad c_0 = 1/2$ $\quad d_0 = -0.35775$

$a_1 = -1/2$ $\quad b_1 = \frac{2.5714}{4}$ $\quad c_1 = 0.14285$

$Q(t) = a_0 + [b_0 + (c_0 + d_0 t)(t-1)]t$

$= -1 + [1/2 + (1/2 - 0.35775t)(t-1)]t$

$Q(0.4) = -0.8857$

④

a) $I = \int_0^2 e^{4x}\,dx \quad f(x) = e^{4x}$

exact solution

$$I = \left[\frac{1}{4}e^{4x}\right]_0^2 = \frac{1}{4}\left(e^8 - e^0\right) = \frac{1}{4}\left(e^8 - 1\right) \approx 744.99$$

Simpson rule $h = 1$

$$S(1) = \frac{h}{3}\left(f(0) + 4f(1) + f(2)\right)$$

$$= \frac{1}{3}\left(e^0 + 4e^4 + e^8\right)$$

$$\approx 1066.783 \qquad \text{absolute error} = 321.793 \ (43.19\%)$$

b) Simpson rule $h = 1/2$

$$S(1/2) = \frac{1}{6}\left(f(0) + 4f(1/2) + 2f(1) + 4f(3/2) + f(2)\right)$$

$$= \frac{1}{6}\left(e^0 + 4e^2 + 2e^4 + 4e^6 + e^8\right)$$

$$\approx 789.071 \qquad \text{absolute error} = 44.081 \ (5.920\%)$$

c) Simpson rule $O(h^4)$ error

$$S(h) = I + ch^4 + O(h^8)$$

$$S\left(\frac{h}{2}\right) = I + c\left(\frac{h}{2}\right)^4 + O(h^8)$$

$$\frac{16\,S(h/2) - S(h)}{15} \approx I$$

$$= 770.556 \qquad \text{absolute error} = 25.566 \ (3.43\%)$$

$$\left(z_1^{j+1}\right)^2 + \left(z_2^{j+1}\right)^2 = \frac{16}{(4+h)^2}\left(\left(1 + \frac{h^2}{2} + \frac{h^4}{16}\right)\left(z_1^{j}\right)^2 + \left(1 + \frac{h^2}{2} + \frac{h^4}{16}\right)\left(z_2^{j}\right)^2\right)$$

$$= \left(z_1^{j}\right)^2 + \left(z_2^{j}\right)^2$$

The circle is preserved.

(5)

a)

$$\ddot{x} + x = 0 \qquad x(0) = x_0 \ , \ \dot{x}(0) = y_0$$

$$z_1 = x$$

$$z_2 = \dot{x} = \dot{z}_1$$

$$\dot{z}_2 = (\ddot{z}_1) = \ddot{x} = -x = -z_1$$

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \end{pmatrix} = \underbrace{\begin{pmatrix} z_2 \\ -z_1 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}_{\underline{\underline{A}}} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}}_{f(t, \underline{z})} \quad \text{with } \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \underline{z}$$

$$\underline{z}^{j+1} = \underline{z}^j + \frac{h}{2}\left(\underline{\underline{A}}\, \underline{z}^j + \underline{\underline{A}}\, \underline{z}^{j+1}\right)$$

$$\left(\underline{\underline{I}} - \frac{h}{2}\underline{\underline{A}}\right) \underline{z}^{j+1} = \underline{z}^j + \frac{h}{2}\underline{\underline{A}}\, \underline{z}^j$$

$$\underline{z}^{j+1} = \left(\underline{\underline{I}} - \frac{h}{2}\underline{\underline{A}}\right)^{-1} \left(\underline{\underline{I}} + \frac{h}{2}\underline{\underline{A}}\right) \underline{z}^j$$

$$= \begin{pmatrix} 1 & -h/2 \\ h/2 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & h/2 \\ -h/2 & 1 \end{pmatrix} \underline{z}^j$$

$$= \frac{1}{1 + (h^2/4)} \begin{pmatrix} 1 & h/2 \\ -h/2 & 1 \end{pmatrix} \begin{pmatrix} 1 & h/2 \\ -h/2 & 1 \end{pmatrix} \underline{z}^j$$

$$= \frac{4}{4 + h^2} \begin{pmatrix} 1 - h^2/4 & h \\ -h & 1 - \frac{h^2}{4} \end{pmatrix} \underline{z}^j$$

b) $R = (z_1^j)^2 + (z_2^j)^2$ check that this is the same for $(z_1^{j+1})^2 + (z_2^{j+1})^2$

$$(z_1^{j+1})^2 = \frac{16}{(4+h^2)^2}\left(\left(1 - \frac{2h^2}{4} + \frac{h^4}{16}\right)(z_1^j)^2 + \left(2h - \frac{h^3}{2}\right)(z_1^j)(z_2^j) + h^2 (z_2^j)^2\right)$$

$$(z_2^{j+1})^2 = \frac{16}{(4+h^2)^2}\left(h^2 (z_1^j)^2 + \left(-2h + \frac{h^3}{2}\right)(z_1^j)(z_2^j) + \left(1 - \frac{2h^2}{4} + \frac{h^4}{16}\right)(z_2^j)^2\right)$$

b) exact solution

$u(x) = a\cos x + b\sin x$

$u(0) = 1$ and $u(1) = 0$

$a = 1$ and $b = -\frac{\cos 1}{\sin 1} = -0.6421$

$u(1/2) = 0.56975$

$h = 1/2$ error $= 1.7\times10^{-3}$ rel. error 3×10^{-3}

$h = 1/4$ error $= 4\times10^{-4}$ rel. error 7.2×10^{-4}