Numerische Mathematik

Prüfung Herbst 2002

1) a) $\delta(a \cdot b) = \frac{\Delta(ab)}{ab} = \frac{\tilde{a}\tilde{b} - ab}{ab} = \frac{(a+\Delta a)(b+\Delta b) - ab}{ab}$

$= \frac{a\,\Delta(b) + b\,\Delta(a) + \Delta(a)\,\Delta(b)}{ab} = \frac{\Delta(b)}{b} + \frac{\Delta(a)}{a} + \frac{\Delta(a)}{a} \cdot \frac{\Delta(b)}{b}$

$= \delta b + \delta a + \delta a \cdot \delta b \leq |\delta(b)| + |\delta(a)| + |\delta(a)|\,|\delta(b)|$

$= (\alpha + \beta)10^{-3} + \alpha\beta 10^{-6}$

ii) $\delta(a/b) = \frac{b}{a}\Delta\left(\frac{a}{b}\right) = \frac{b}{a}\left(\frac{\tilde{a}}{\tilde{b}} - \frac{a}{b}\right) = \frac{b}{a}\left(\frac{a+\Delta(a)}{b+\Delta(b)} - \frac{a}{b}\right) = \frac{b}{a}\,\frac{b\,\Delta(a) - a\,\Delta(b)}{b(b+\Delta(b))}$

$= \frac{b}{a}\,\frac{ba\,\delta(a) - ab\,\delta(b)}{b(b + b\,\delta(b))} = \frac{\delta(a) - \delta(b)}{1 + \delta(b)} \leq \frac{|\delta(a)| + |\delta(b)|}{1 - |\delta(b)|}$

$= \frac{(\alpha+\beta)10^{-3}}{1 - \beta 10^{-3}}$

b) $\sqrt{a} - \sqrt{b}$ gives error for $a \approx b$

Here $a - b = 1$ and $a, b \gg 1$

$\sqrt{a} - \sqrt{b} = (\sqrt{a} - \sqrt{b})\frac{(\sqrt{a} + \sqrt{b})}{\sqrt{a} + \sqrt{b}} = \frac{a-b}{\sqrt{a}+\sqrt{b}} = \frac{1}{35136.4123 - 35136.41229}$

$= \frac{1}{70272.82359} = 1.42302495 \times 10^{-5}$

c) $A = \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}$ Note matrix not symmetric!

$\kappa_2(A, \|\cdot\|_2) = \|A\|_2\,\|A_2^{-1}\| = \sqrt{\frac{\mu_M}{\mu_m}}$ $\quad \mu_M = \max \lambda(A^TA)$

$\mu_m = \min \lambda(A^TA)$

$A^TA = \begin{pmatrix} 10 & 5 \\ 5 & 5 \end{pmatrix}$ $\quad (10-\lambda)(5-\lambda) - 25 = 0$ $\quad \lambda_{1,2} = \frac{15 \pm \sqrt{225 - 100}}{2}$

$50 + \lambda^2 - 5\lambda - 10\lambda - 25 = 0$

$25 + \lambda^2 - 15\lambda = 0$ $\quad = 1.9098,\ 13.090$

$\therefore \kappa = 2.6180$

$\frac{\Delta a}{a} = \delta a$

2) $\quad n=3$

$$\underline{\underline{A}} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \qquad \underline{b} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$u_0 = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}$$

$$r_0 = Au + b = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

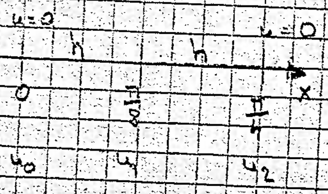
$$p_0 = -r_0 = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} \qquad A p_0 = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

$$\rho = \langle r, r \rangle \Big/ \langle p, Ap \rangle \qquad \rho_0 = \frac{1}{2}$$

$$u_1 = u_0 + \rho p_0 = -\begin{pmatrix} 1 \\ 3/2 \\ 1 \end{pmatrix}$$

3) a) Implicit Euler is unconditionally stable. No stability limit required.

b)



Central differences: $u''(x) \approx \dfrac{u_0 - 2u_1 + u_2}{h^2} = \dfrac{-2u_1}{\left(\frac{\pi}{8}\right)^2} = -\dfrac{128}{\pi^2} u_1 \quad (\approx u_{xx}) = f(t, u_1$

At $x = \frac{\pi}{8}$ $\quad u_1(0) = u(0, x)\Big|_{x=\frac{\pi}{8}} = \sin \frac{4\pi}{8} = \sin \frac{\pi}{2} = 1 = u_1^0$

Impl. Euler:

$u_1(t) = u_1^0 + \Delta t\, f(\Delta t, u_1^1) = 1 + h\left(-\dfrac{128}{\pi^2} u_1\right) = 1 + \dfrac{1}{2}\left(\dfrac{128}{\pi^2} u_1\right)$

$u_1^1 = \dfrac{1}{1 + \frac{64}{\pi^2}} = 0.133608$

4. Non linear system

$$(y-x)x + xy = 1/2$$

$$(x-y)^2 x + xy^2 = 1/3$$

$$F = \begin{pmatrix} (y-x)x + xy - 1/2 \\ (x-y)^2 x + xy^2 - 1/3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = F(x,y)$$

Central Difference approximation for Jacobian $\tilde{J}^0 = \frac{1}{2h}\left(F(x^0+h, y^0) - F(x^0-h, y^0), F(x^0, y^0+h) - F(x, y^0 \ldots\right)$

$$J_{11} = \frac{10}{2}\left((9/10 - 1/10 - 1/10)\, 2/10 + 9/10 \cdot 2/10 - 1/2 - \left((9/10 - 1/10 + 1/10)\, 0/10 + 9/10 \cdot 0/10 - 1/2 \right) \right)$$

$$= \frac{10}{2}\left(14/10^2 + 18/10^2 - 1/2 + 1/2 \right) = 1.6$$

$$J_{21} = \frac{10}{2}\left((2/10 - 9/10)^2\, 2/10 + 2/10 \cdot 9^2/10^2 - 1/3 - \left((-9/10)^2\, 0/10 + 0/10\,(9/10)^2 - 1/3 \right) \right)$$

$$= \frac{10}{2}\left(\frac{(-7)^2}{100} \cdot \frac{2}{10} + \frac{2}{10} \cdot \frac{9^2}{100} \right)$$

$$= \frac{1}{1}\left(\frac{49}{100} + \frac{81}{100} \right) = 1.3$$

$$J_{12} = \frac{10}{2}\left(\left(\frac{9}{10} + \frac{1}{10} - \frac{1}{10}\right)\frac{1}{10} + \frac{10}{10}\frac{1}{10} - \frac{1}{2} - \left(\left(\frac{8}{10} - \frac{1}{10}\right)\frac{1}{10} + \frac{8}{10}\frac{1}{10} - \frac{1}{2} \right) \right)$$

$$= \frac{10}{2}\left(\frac{9}{100} + \frac{10}{100} - \frac{1}{2} - \left(\frac{7}{100} + \frac{8}{100} - \frac{1}{2} \right) \right)$$

$$= \frac{10}{2}\left(\frac{19}{100} - \frac{15}{100} \right) = 0.2$$

$$J_{22} = \frac{10}{2}\left(\left(\frac{1}{10} - \frac{10}{10}\right)^2 \frac{1}{10} + \frac{1}{10}\left(\frac{10}{10}\right)^2 - \frac{1}{3} - \left(\left(\frac{1}{10} - \frac{8}{10}\right)^2 \frac{1}{10} + \frac{1}{10}\left(\frac{8}{10}\right)^2 - \frac{1}{3} \right) \right)$$

$$= \frac{10}{2}\left(\frac{81}{100}\frac{1}{10} + \frac{100}{1000} - \left(\frac{49}{100}\frac{1}{10} + \frac{64}{1000} \right) \right)$$

$$= \frac{1}{2}\left(\frac{81}{100} + \frac{100}{100} - \frac{113}{100} \right) = 0.34$$

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} - \underline{J}^{-1} \underline{F}(x_0, y_0) \qquad \underline{F}(x_0, y_0) = -\begin{pmatrix} 33/100 \\ 113/600 \end{pmatrix}$$

Solve $\underline{J}\,\delta = -\underline{F}$ for δ $\qquad \begin{matrix} 1.6 & 0.2 \\ 1.3 & 0.34 \end{matrix} \begin{pmatrix} 33/100 \\ 113/60 \end{pmatrix} \Rightarrow \delta = \begin{pmatrix} 0.2624 \\ -0.4495 \end{pmatrix}$

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 1/10 \\ 9/10 \end{pmatrix} + \delta = \begin{pmatrix} 0.3624 \\ 0.4505 \end{pmatrix}$$

5a) $I = \int_0^1 1 - \frac{x^2}{1!} + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} \dots + (-1)^k \frac{x^{2k}}{k!} \, dx$

$= x - \frac{x^3}{3 \cdot 1!} + \frac{x^5}{5 \cdot 2!} - \frac{x^7}{7 \cdot 3!} + \frac{x^9}{9 \cdot 4!} \dots + (-1)^k \frac{x^{2k+1}}{(2k+1)k!} \Big|_0^1$

$= 1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} + \frac{1}{216} - \frac{1}{1320} + \frac{1}{9360} = 0.747487$

$|\cdot| < 10^{-3}$ $\quad = 0.747 =: I_{a)}$

b) $I(1) = 1/2\,(e^{-0} + e^{-1}) = 0.6839 = I + O(h^2)$

$I(1/2) = 1/4\,(e^{-0} + 2e^{-1/4} + e^{-1}) = 0.73137 = I + O((h/2)^2)$

$I \approx \frac{R_{10} - 4^{-1} R_{00}}{1 - 4^{-1}} = \frac{4R_{10} - R_{00}}{4 - 1} = \frac{4R_{10} - R_{00}}{3}$

$R_{10} = I(1/2) \qquad R_{00} = I(1)$

$I = \frac{4I(1/2) - I(1)}{3} = 0.74718 =: I_{b)}$

For a) we know that $|I - I_{a)}| < 10^{-3}$

and that $|I_{a)} - I_{b)}| \approx 3 \times 10^{-4} < 10^{-3}$

Therefore b) is also correct to a relative error of $< 10^{-3}$.

6a) 3rd order ODE 3 conditions required

$x(0) = 1$, $x'(0) = \alpha$, $x''(0) = \beta$

b) $y^1 = x$ $\quad (y^1)' = y^2$

$y^2 = x'$ $\quad (y^2)' = y^3$

$y^3 = x''$ $\quad (y^3)' = -x' - x = -y^2 - y^1$

$$(\underline{y})' = \underbrace{\overbrace{\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & 0 \end{pmatrix}}^{\underline{\underline{A}}} \begin{pmatrix} y^1 \\ y^2 \\ y^3 \end{pmatrix}}_{:= f(\underline{y})} \qquad \text{with } \underline{y}(0) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

...ge-Kutta

- $K_1 = f(y + \frac{h}{2} K_1) = A(\underline{y} + \frac{h}{2} K_1) = \underline{\underline{A}}\,\underline{y} + \frac{h}{2} \underline{\underline{A}}\,\underline{K}_1$

$$(\underline{\underline{I}} - \tfrac{h}{2} \underline{\underline{A}}) \underline{K}_1 = \underline{\underline{A}}\,\underline{y}$$

- $K_2 = f(y + \frac{h}{2} K_1) = \underline{\underline{A}}\,\underline{y} + \frac{h}{2} \underline{\underline{A}} K_1 = K_1$

Iteration $\quad y_{n+1} = y_n + \frac{h}{2}(K_1 + K_2) = y_n + h K_1$

Calculate K_1, $\quad (\underline{\underline{I}} - \frac{h}{2}\underline{\underline{A}}) K_1 = \underline{\underline{A}}\,\underline{y} \qquad h = 0.2$

$$(\underline{\underline{I}} - 0.1 \underline{\underline{A}}) K_1 = \underline{\underline{A}}\,\underline{y}$$

$$\begin{pmatrix} 1 & -1/10 & 0 \\ 0 & 1 & -1/10 \\ 1/10 & 1/10 & 1 \end{pmatrix} \begin{pmatrix} K_1^1 \\ K_1^2 \\ K_1^3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$$

$K_1^3 = -2.18595$

$K_1^2 = 0.7814048$

$K_1^1 = 1.07814046$

Solution is $x(t) = y^1(t)$

$y^1(0.2) \approx y^1(0) + h K_1^1 = 1.2156$