· ∴Herbst exam 2003

① $A = D + L + R$

$D = \begin{pmatrix} a & 0 \\ 0 & 4 \end{pmatrix}$ $L = \begin{pmatrix} 0 & 0 \\ 3 & 0 \end{pmatrix}$ $R = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$

$Ax = (D+L+R)x = b$

Jacobi: $D x^{(n+1)} = -(L+R)x^{(n)} + b \Rightarrow T_J = -D^{-1}(L+R) = -\begin{pmatrix} 1/a & 0 \\ 0 & 1/4 \end{pmatrix}\begin{pmatrix} 0 & 2 \\ 3 & 0 \end{pmatrix} = -\begin{pmatrix} 0 \\ 3/4 \end{pmatrix}$

G.S. : $(D+L)x^{(n+1)} = -Rx^{(n)} + b \Rightarrow T_{GS} = -(D+L)^{-1}R = -\begin{pmatrix} a & 0 \\ 3 & 4 \end{pmatrix}^{-1}\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -2/a \\ 0 & 3/2a \end{pmatrix}$

$a \neq 0$ D regular, $(D+L)$ regular convergence if $\rho(T) < 1$

a) i) $\det\left[\begin{pmatrix} -\lambda & -2/a \\ -3/4 & -\lambda \end{pmatrix}\right] = \lambda^2 - \frac{6}{4a} \Rightarrow \lambda^2 = \frac{3}{2a}$

$a > 0$: $\lambda_i = \pm\sqrt{\frac{3}{2a}}$ $\rho(T) = \sqrt{\frac{3}{2a}} < 1 \Rightarrow \sqrt{a} > \sqrt{\frac{3}{2}} \Rightarrow a > \frac{3}{2}$ $(a>0)$

ii) $\lambda_1 = 0$, $\lambda_2 = \frac{3}{2a}$ $|\lambda_2| < 1$ $a > 0$; $a > \frac{3}{2}$

b) $x^{(1)} = -(D+L)^{-1}R\begin{pmatrix} 1 \\ 1 \end{pmatrix} + (D+L)^{-1}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$= \begin{pmatrix} 0 & -2/5 \\ 0 & 3/10 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1/5 & 0 \\ -3/20 & 1/4 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$= \begin{pmatrix} -2/5 \\ 3/10 \end{pmatrix} + \begin{pmatrix} 0 \\ 1/4 \end{pmatrix} = \begin{pmatrix} -2/5 \\ 11/20 \end{pmatrix}$

② $\left| \begin{array}{l} 16 + 9 + 4a + 3b + c = 0 \\ 9 + 36 + 3a + 6b + c = 0 \\ 4 + 1 + 2a + b + c = 0 \\ 0 + 36 + 0a + 6b + c = 0 \end{array} \right|$

$$\underbrace{\begin{pmatrix} 4 & 3 & 1 \\ 3 & 6 & 1 \\ 2 & 1 & 1 \\ 0 & 6 & 1 \end{pmatrix}}_{\underline{\underline{A}}} \underbrace{\begin{pmatrix} a \\ b \\ c \end{pmatrix}}_{\underline{x}} = \underbrace{\begin{pmatrix} -25 \\ -45 \\ -5 \\ -36 \end{pmatrix}}_{\underline{b}}$$

$\underline{\underline{A}}\,\underline{x} - b = \underline{r}$

$\|r\|^2 \overset{!}{=} \min$

$$\Rightarrow A^T A x = \overbrace{\begin{pmatrix} 29 & 32 & 9 \\ 32 & 82 & 16 \\ 9 & 16 & 4 \end{pmatrix}}^{\tilde{A}} \underline{X}$$

$$A^T b = -\begin{pmatrix} 100 + 135 + 10 \\ 75 + 270 + 5 + 216 \\ 25 + 45 + 5 + 36 \end{pmatrix} = -\underbrace{\begin{pmatrix} 245 \\ 566 \\ 111 \end{pmatrix}}_{\tilde{b}}$$

Solve $\tilde{A} x = \tilde{b}$ with LU of $\tilde{A}$

$LUx = \tilde{b}$ solve $Ly = \tilde{b}$ $Ux = y$

(Show working)

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 32/29 & 1 & 0 \\ 9/29 & 88/677 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 29 & 32 & 9 \\ 0 & 1354/29 & 176/29 \\ 0 & 0 & 8207/19633 \end{pmatrix}$$

$$Ly = \tilde{b} \Rightarrow y = \begin{pmatrix} -245 \\ -8574/29 \\ 68034/19633 \end{pmatrix}$$

$$Ux = y \Rightarrow x = \begin{pmatrix} -2.8445 \\ -7.4099 \\ 8.2898 \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

(show working)

③ $F(x) = \frac{1}{1+x}$ $x > -1$ $F \in C^1(I)$ $\forall I \subset (-1, \infty)$

a) For BFPS

i) Kontraction : $F \in C^1$, $L = \sup_{x \in I} |F'(x)| = \sup_{x \in I} \frac{1}{(1+x)^2} < 1$

$\Rightarrow x > 0$ therefore $I \subset (\varepsilon, \infty)$, $\varepsilon > 0$

ii) Selbstabb : $F' < 0$ therefore F monton ↓ for $x \uparrow$ $(x > 0) \Rightarrow$

$Im(F(x)) \subset [0, 1)$ for $x > 0$ $\left(\lim_{x \to \infty} F(x) = 0, \lim_{x \to 0} F(x) = 1 \right)$

$\Rightarrow I = [a, b]$ with $a > 0$ arbitrary small, $b < \infty$ arbitrary large

Because $\lim_{x \to 0} F(x) = 1$, then $b \geq 1$ for arbitrary small $a > 0$

b) $|\text{error}| \leq \frac{L^k}{1-L} |x_1 - x_0| \leq 10^{-2}$

$L = \sup_{x \in [\frac{1}{10}, 10]} |F'(x)| = \frac{1}{(1 + \frac{1}{10})^2} = \frac{100}{121} \approx 0.8246$

$|x_1 - x_0| = |F(x_0) - x_0| = \left| \frac{1}{1+9} - 9 \right| = 8.9$

$L^k \leq 10^{-2} \frac{1-L}{|x_1 - x_0|}$

$k \geq \frac{\log\left(10^{-2} \left(\frac{1-L}{|x_1 - x_0|}\right)\right)}{\log L} = 44.87 \Rightarrow$ at least 45 iteration.

(4)

a) $T(\frac{1}{2}) = \frac{1}{2}\left(\frac{1}{2}e^{-\lambda(0)^2} + e^{-\lambda(1/2)^2} + \frac{1}{2}e^{-\lambda 1^2}\right)$

$= 1/4 + 1/2 e^{-\lambda/4} + 1/4 e^{-\lambda}$

b) $F(\lambda) = T(1/2) - 1/2 = 0 = -1/4 + 1/2 e^{-\lambda/4} + 1/4 e^{-\lambda}$

$F'(\lambda) = -1/8\, e^{-\lambda/4} - 1/4 e^{-\lambda}$

Newton

$$\lambda_1 = \lambda_0 - \frac{F(\lambda_0)}{F'(\lambda_0)} = 3 - \frac{-\frac{1}{4} + \frac{1}{2}e^{-3/4} + \frac{1}{4}e^{-3}}{-\frac{1}{8}e^{-3/4} - \frac{1}{4}e^{-3}}$$

≈ 2.9808

(5)

$$\ddot{y}_1 = 2y_1 - 3y_2 \qquad \text{set } x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} := \begin{pmatrix} y_1 \\ y_2 \\ \dot{y}_1 \\ \dot{y}_2 \end{pmatrix}$$

$$\ddot{y}_2 = -3y_1 + 4y_2$$

$$\dot{x}_1 = x_3$$
$$\dot{x}_2 = x_4$$
$$\dot{x}_3 = 2x_1 - 3x_2$$
$$\dot{x}_4 = -3x_1 + 4x_2$$

$$\dot{x} = \underbrace{\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & -3 & 0 & 0 \\ -3 & 4 & 0 & 0 \end{pmatrix}}_{=:A} x \quad , \quad x(0) = \begin{pmatrix} 2 \\ 1 \\ 0 \\ 1 \end{pmatrix} = x_0$$

- trapezoidal method for $\dot{x} = Ax$

$$x_1 = x_0 + \frac{h}{2}(Ax_0 + Ax_1) \qquad h = 1$$

$$\Rightarrow x_1 = \left(I - \frac{1}{2}A\right)^{-1}\left(I + \frac{1}{2}A\right)x_0$$

$$\begin{pmatrix} 1 & 0 & -1/2 & 0 \\ 0 & 1 & 0 & -1/2 \\ -1 & 3/2 & 1 & 0 \\ 3/2 & -2 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1/2 & 0 \\ 0 & 1 & 0 & 1/2 \\ 1 & -3/2 & 1 & 0 \\ -3/2 & 2 & 0 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 3/2 \\ 1/2 \\ 0 \end{pmatrix}$$

- $$\left.\begin{matrix} 1 & 0 & -1/2 & 0 \\ 0 & 1 & 0 & -1/2 \\ -1 & 3/2 & 1 & 0 \\ 3/2 & -2 & 0 & 1 \end{matrix}\;\right|\; \begin{matrix} 2 \\ 3/2 \\ 1/2 \\ 0 \end{matrix} \quad 1 \quad -3/2 \qquad \text{gives} \qquad \left.\begin{matrix} 1 & 0 & -1/2 & 0 \\ 0 & 1 & 0 & -1/2 \\ 0 & 3/2 & 1/2 & 0 \\ 0 & -2 & 3/4 & 1 \end{matrix}\;\right|\; \begin{matrix} 2 \\ 3/2 \\ 5/2 \\ -3 \end{matrix} \quad -3/2 \quad 2$$

$$\left.\begin{matrix} 1 & 0 & -1/2 & 0 \\ 0 & 1 & 0 & -1/2 \\ 0 & 0 & 1/2 & 3/4 \\ 0 & 0 & 3/4 & 0 \end{matrix}\;\right|\; \begin{matrix} 2 \\ 3/2 \\ 1/4 \\ 0 \end{matrix} \quad -3/2 \qquad \text{gives} \qquad \left.\begin{matrix} 1 & 0 & -1/2 & 0 \\ 0 & 1 & 0 & -1/2 \\ 0 & 0 & 1/2 & 3/4 \\ 0 & 0 & 0 & -9/8 \end{matrix}\;\right|\; \begin{matrix} 2 \\ 3/2 \\ 1/4 \\ -3/8 \end{matrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2 \\ 5/3 \\ 0 \\ 1/3 \end{pmatrix}$$

$$y_1(1) \approx 2$$
$$y_2(1) \approx 5/3$$

b) a)

$$K_1 = f\left(t_n + \frac{h}{2}, x_n + h\gamma K_1\right)$$

$$x_{n+1} = x_n + hK_1$$

b) $f(t,x) = \lambda x \ , \ \lambda \in \mathbb{C}$

$$K_1 = \lambda(x_n + h\gamma K_1) = \lambda x_n + h\lambda\gamma K_1 \Rightarrow K_1 = \frac{\lambda x_n}{1 - h\lambda\gamma}$$

$$x_{n+1} = x_n + x_n \frac{\lambda}{1-h\lambda\gamma}$$

$$= x_n\left(1 + \frac{\mu}{1-\mu\gamma}\right) = x_n \underbrace{\left(\frac{1-(\gamma-1)\mu}{1-\mu\gamma}\right)}_{R(\mu)} \quad \text{with } \mu = h\lambda$$

c) $|R| = 1 \Leftrightarrow |R|^2 = 1$

$|R|^2 = RR^*$

Set $\mu = x + iy$ then $RR^* = \dfrac{1-(\gamma-1)x-(\gamma-1)iy}{1-\gamma x - \gamma iy} \; \dfrac{1-(\gamma-1)+(\gamma-1)iy}{1-\gamma x + \gamma iy}$

we want $|R| = 1 \quad \forall y \in \mathbb{R}$ with $x = 0$:

$$RR^*\big|_{x=0} = \frac{(1-(\gamma-1)iy)(1+(\gamma-1)iy)}{(1-\gamma iy)(1+\gamma iy)} = \frac{(1+(\gamma-1)^2y^2)}{1+\gamma y^2} \overset{!}{=} 1$$

$$(\gamma-1)^2 = \gamma^2 \Rightarrow \underline{\underline{\gamma = 1/2}}$$

d) $S(\mu) = \{\mu \in \mathbb{C} : |R(\mu)| < 1\}$

$$\gamma = 1/2 : R(\mu) = \frac{1-\mu/2}{1+\mu/2} = \frac{2-\mu}{2+\mu}$$

Like Trapozoidal method $\Rightarrow S(\mu) = \{\mu \in \mathbb{C} : \operatorname{Re}\mu < 0\}$