1.

$$A = \begin{bmatrix} 0{,}01 & 2 & 0 \\ 2 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} , \quad b = \begin{bmatrix} 0{,}1e^{1} \\ 0{,}1e^{1} \\ 0 \end{bmatrix}$$

i)

$$A = \begin{bmatrix} 0{,}1e^{-1} & 0{,}2e^{1} & 0 \\ 0{,}2e^{1} & 0{,}2e^{1} & 0{,}1e^{1} \\ 0 & 0{,}1e^{1} & 0{,}2e^{1} \end{bmatrix}$$

$$\frac{0{,}2e^{1}}{0{,}1e^{-1}} = \frac{2e^{0}}{1e^{-2}} = 2e^{2} = 0{,}2e^{3} \qquad 0{,}2e^{1} - 0{,}2e^{3} \cdot 0{,}2e^{1} = -0{,}4e$$

$$\xrightarrow{Z_2 - 0{,}2e^{3} Z_1} \begin{bmatrix} 0{,}1e^{-1} & 0{,}2e^{1} & 0 \\ 0 & -0{,}4e^{3} & 0{,}1e^{1} \\ 0 & 0{,}1e^{1} & 0{,}2e^{1} \end{bmatrix}$$

$$\frac{0{,}1e^{1}}{0{,}4 \cdot 10^{3}} = 0{,}25 \cdot 10^{-2} = 0{,}3e^{-2}$$

$$\xrightarrow{Z_3 + 0{,}3e^{-2} Z_2} \begin{bmatrix} 0{,}1e^{-1} & 0{,}2e^{1} & 0 \\ 0 & -0{,}4e^{3} & 0{,}1e^{1} \\ 0 & 0 & 0{,}2e^{1} \end{bmatrix}$$

$$0{,}2e^{1} + 0{,}3e^{-2} \cdot 0{,}1e^{1} = 0{,}2e^{1}$$

$$\Longrightarrow R = \begin{bmatrix} 0{,}1e^{-1} & 0{,}2e^{1} & 0 \\ 0 & -0{,}4e^{3} & 0{,}1e^{1} \\ 0 & 0 & 0{,}2e^{1} \end{bmatrix}$$ 1 Pt for R.

$$\Longrightarrow L = \begin{bmatrix} 0{,}1e^{1} & 0 & 0 \\ +0{,}2e^{3} & 0{,}1e^{1} & 0 \\ 0 & -0{,}3e^{-2} & 0{,}1e^{1} \end{bmatrix}$$ 1 Pt for L.

$. LR \cdot \approx A$, $Ax = b$ $\Longrightarrow$ $L\underbrace{(Rx)}_{=:c} \approx b$

Solve $Lc = b$:

$\Longrightarrow L_{11} c_1 = b_1 \Longrightarrow 0{,}1e^1 c_1 = 0{,}1e^1$

$\Longrightarrow c_1 = 0{,}1e^1$

$\Longrightarrow L_{21} c_1 + L_{22} c_2 = b_2 \Longrightarrow +0{,}2e^3 \cdot 0{,}1e^1 + 0{,}1e^1 \cdot c_2 = 0{,}1e$

$\Longrightarrow 0{,}1e^1 \cdot c_2 = 0{,}1e^1 - 0{,}2e^3$

$= -0{,}2e^3$

$\Longrightarrow c_2 = -0{,}2e^3$

$\Longrightarrow L_{31} c_1 + L_{32} c_2 + L_{33} c_3 = b_3 \Longrightarrow 0 \cdot 0{,}1e^1 - 0{,}3e^{-2}(-0{,}2e^3) + 0{,}1e^1 c_3 = 0$

$\Longrightarrow 0{,}1e^1 c_3 = -0{,}6e^0$

$\Longrightarrow c_3 = -0{,}6e^0$

$$\Longrightarrow c = \begin{pmatrix} 0{,}1e^1 \\ -0{,}2e^3 \\ -0{,}6e^0 \end{pmatrix}$$

1 Pt for the solution of $Lc = b$

Solve $Rx = c$:

$\Longrightarrow R_{33} x_3 = c_3 \Longrightarrow 0{,}2e^1 x_3 = -0{,}6e^0$

$\Longrightarrow x_3 = -0{,}3e^0$

$$\Rightarrow R_{22}x_2 + R_{23}x_3 = c_2 \Rightarrow -0{,}4e^3 x_2 + 0{,}1e^1(-0{,}3e^0) = -0{,}2e^3$$

$$\Rightarrow -0{,}4e^3 x_2 = -0{,}2e^3 + 0{,}3e^0$$

$$\Rightarrow x_2 = \frac{-0{,}2e^3}{-0{,}4e^3}$$

$$= \frac{0{,}2}{0{,}4} e^0$$

$$= 0{,}5e^0$$

$$\Rightarrow R_{11}x_1 + R_{12}x_2 + R_{13}x_3 = c_1 \Rightarrow 0{,}1e^{-1}x_1 + 0{,}2e^1(\cdot 0{,}5e^0) + 0 \cdot (-0{,}3e$$

$$= 0{,}1e^1$$

$$\Rightarrow 0{,}1e^{-1}x_1 = 0{,}1e^1 + 0{,}1e^0$$

$$\Rightarrow x_1 = \frac{0{,}1e^0}{0{,}1e^{-1}}$$

$$= 0{,}1e^1$$

$$\Rightarrow x = \begin{pmatrix} 0{,}1e^2 \\ 0{,}5e^0 \\ -0{,}3e^0 \end{pmatrix}$$

1 Pt for the solution of $Rx = c$.

ii) $A' = \begin{bmatrix} 0,1e^{-1} & 0,2e^{1} & 0 \\ 0,2e^{1} & 0,2e^{1} & 0,1e^{1} \\ 0 & 0,1e^{1} & 0,2e^{1} \end{bmatrix}$

$$\xrightarrow{Z_1 \leftrightarrow Z_2} \begin{bmatrix} 0,2e^{1} & 0,2e^{1} & 0,1e^{1} \\ 0,1e^{-1} & 0,2e^{1} & 0 \\ 0 & 0,1e^{1} & 0,2e^{1} \end{bmatrix}$$

$$\frac{0,1e^{-1}}{0,2e^{1}} = 0,5e^{-2}$$

$0,2e^{1} - 0,5e^{-2} \cdot 0,2e^{1} = 0,2e^{1} - 0,1e^{-1} = 0,2$

$$\xrightarrow{Z_2 - 0,5e^{-2} Z_1} \begin{bmatrix} 0,2e^{1} & 0,2e^{1} & 0,1e^{1} \\ 0 & 0,2e^{1} & -0,5e^{-2} \\ 0 & 0,1e^{1} & 0,2e^{1} \end{bmatrix}$$

$0 - 0,5e^{-2} \cdot 0,1e^{1} = -0,5e$

$$\frac{0,1e^{1}}{0,2e^{1}} = 0,5e^{0}$$

$$\xrightarrow{Z_3 - 0,5e^{0} Z_2} \begin{bmatrix} 0,2e^{1} & 0,2e^{1} & 0,1e^{1} \\ 0 & 0,2e^{1} & -0,5e^{-2} \\ 0 & 0 & 0,2e^{1} \end{bmatrix}$$

$0,2e^{1} - 0,5e^{0}(-0,5e^{-2}) = 0,2e^{1} + 0,3e$
$= 0,2e^{1}$

$$\Rightarrow \quad P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \quad L = \begin{bmatrix} 0,1e^{1} & 0 & 0 \\ +0,5e^{-2} & 0,1e^{1} & 0 \\ 0 & +0,5e^{0} & 0,1e^{1} \end{bmatrix}$$ 1 Pt. for L

$$\Rightarrow \quad R = \begin{bmatrix} 0,2e^{1} & 0,2e^{1} & 0,1e^{1} \\ 0 & 0,2e^{1} & -0,5e^{-2} \\ 0 & 0 & 0,2e^{1} \end{bmatrix}$$ 1 Pt. for R

$$\Longrightarrow \quad PAx = Pb \quad \Rightarrow \quad Pb = \begin{bmatrix} 0{,}1e^1 \\ 0{,}1e^1 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad L\underbrace{(Rx)}_{=:c} \approx Pb$$

Solve $\quad Lc = Pb$

$$\Rightarrow L_{11}\, c_1 = (Pb)_1 \Rightarrow 0{,}1e^1\, c_1 = 0{,}1e^1$$

$$\Rightarrow \quad c_1 = 0{,}1e^1$$

$$\Rightarrow L_{21} c_1 + L_{22} c_2 = (Pb)_2 \Rightarrow 0{,}5e^{-2}\, 0{,}1e^1 + 0{,}1e^1 c_2 = 0{,}1e$$

$$\Rightarrow c_2 = 0{,}1e^1 - 0{,}5e^{-2}$$

$$= 0{,}1e^1$$

$$\Rightarrow L_{31}c_1 + L_{32}c_2 + L_{33}c_3 = (Pb)_3 \Rightarrow 0 + 0{,}5e^0 \cdot 0{,}1e^1 + 0{,}1e^1 c_3 = 0$$

$$\Rightarrow \quad c_3 = -0{,}5e^0$$

$$\Rightarrow \quad c = \begin{pmatrix} 0{,}1e^1 \\ 0{,}1e^1 \\ -0{,}5e^0 \end{pmatrix}$$ 1 Pt. for the solution of $Lc = Pb$

Solve $\quad Rx = c$:

$$\rightarrow \quad R_{33}\, x_3 = c_3 \quad \Rightarrow \quad 0{,}2e^1\, x_3 = -0{,}5e^0$$

$$\Rightarrow \quad x_3 = -\frac{0{,}5e^0}{0{,}2e^1} = -0{,}25e^0$$

$$= -0{,}3e^0$$

$$\Rightarrow R_{22}x_2 + R_{23}x_3 = c_2 \Rightarrow 0{,}2e^1 x_2 + 0{,}5e^{-2}(-0{,}3e^0) = 0{,}1e$$

$$\Rightarrow x_2 = \frac{0{,}1e^1 + 0{,}2e^{-2}}{0{,}2e^1}$$

$$= 0{,}5e^0$$

$$\Rightarrow R_{11}x_1 + R_{12}x_2 + R_{13}x_3 = c_3 \Rightarrow 0{,}2e^1 x_1 + 0{,}2e^1 \cdot 0{,}5e^0 - 0{,}1e^1 \cdot 0{,}3e$$

$$= 0{,}1e^1$$

$$\Rightarrow x_1 = \frac{0{,}1e^1 + 0{,}3e^0 - 0{,}1e}{0{,}2e^1}$$

$$= \frac{0{,}3e^0}{0{,}2e^1}$$

$$= 0{,}2e^0$$

$$\Rightarrow x = \begin{pmatrix} 0{,}2e^0 \\ 0{,}5e^0 \\ -0{,}3e^0 \end{pmatrix}$$

1 Pt. for the solution of

$$Rx = c.$$

$$\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4,3636 \\ -0,7967 \end{pmatrix}$$

3 Pts. for correct solution.

$\Rightarrow$ $k = -0,7967$, $M_0 = e^x = 78,5378$

1 Pt for correct k and M_0.

Σ 8

3\. a) Skript Seite 52:

$$f(x) - P_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} \prod_{i=0}^{n} (x - x_i)$$

$$\Rightarrow \quad n = 2: \quad |f(x) - P_2(x)| = \frac{|f^{(3)}(\xi)|}{3!} \prod_{i=0}^{2} |x - x_i|$$

$$\leq \frac{M_3}{6} |x - x_1||x - x_2||x - x_3|$$

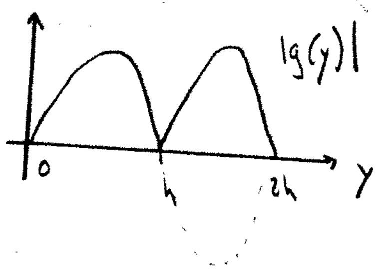
$$= \frac{M_3}{6} |x - x_0||x - x_0 - h||x - x_0 - 2h|$$

① Pt for correct upper bound

$$\max_{x \in [x_0, x_0+2h]} |x - x_0||x - x_0 - h||x - x_0 - 2h|$$

$$= \max_{y \in [0, 2h]} |y||y - h||y - 2h|$$

$$\Rightarrow \quad g(y) = y(y-h)(y-2h),$$



$$\Rightarrow \quad g(y) = y(y^2 - 3hy + 2h^2)$$

$$= y^3 - 3y^2h + 2yh^2$$

① Pt. for minimization function.

$$\rightarrow \quad \frac{\partial g}{\partial y}(y) = 3y^2 - 6yh + 2h^2$$

(1

$$\Rightarrow \frac{\partial g}{\partial y} = 0 \iff 3y^2 - 6yh + 2h^2 = 0$$

$$\Rightarrow y_{1/2} = \frac{6h \pm \sqrt{36h^2 - 4 \cdot 3 \cdot 2h^2}}{6}$$

$$\Rightarrow y_{1/2} = h \pm \frac{\sqrt{12}}{6} h$$

$$= h \cdot (1 \pm 1/\sqrt{3})$$ ① Pt. for correct minim

$$\Rightarrow |g(y)| = |y|\,|y-h|\,|y-2h|$$

$$= |1 \pm 1/\sqrt{3}|\,|\pm 1/\sqrt{3}|\,|-1 \pm 1/\sqrt{3}|\, h^3$$

$$= |1 \pm 1/\sqrt{3}|\; 1/\sqrt{3}\; |1 \mp 1/\sqrt{3}|\, h^3$$

$$= \left|\left(1 + 1/\sqrt{3}\right) 1/\sqrt{3} \left(1 - 1/\sqrt{3}\right)\right| h^3$$

$$= \left|\left(1 - 1/3\right) 1/\sqrt{3}\right| h^3$$

$$= \frac{2}{3} \frac{h^3}{\sqrt{3}}$$

$$\Rightarrow |f(x) - P_2(x)| \leq \frac{M_3}{6} \frac{2}{3} \frac{h^3}{\sqrt{3}} = \frac{2}{18\sqrt{3}} M_3 h^3$$

① Pt. for correct upper bound.

b) $\cdot$ $P_n(x) = \sum_{i=0}^{n} l_i(x) f_i$ mit $l_i(x) = \prod_{\substack{j=0 \\ j\neq i}}^{n} \frac{(x-x_j)}{(x_i - x_j)}$ (11)

① Pt. for the definition of interpol. polynomia[l]

$$\Rightarrow l_0(x) = \frac{x-x_1}{x_0-x_1} \cdot \frac{x-x_2}{x_0-x_2} \cdot \frac{x-x_3}{x_0-x_3}$$

$$= \frac{(x-2)}{-1} \frac{(x-3)}{-2} \frac{(x-4)}{-3}$$

$$= -\frac{1}{6}(x-2)(x-3)(x-4)$$

$$\Rightarrow l_1(x) = \frac{(x-x_0)}{(x_1-x_0)} \frac{(x-x_2)}{(x_1-x_2)} \frac{(x-x_3)}{(x_1-x_3)}$$

$$= \frac{(x-1)}{1} \frac{(x-3)}{-1} \frac{(x-4)}{-2}$$

$$= \frac{1}{2}(x-1)(x-3)(x-4)$$

② Pts. for corre[ct] $l_0(x), l_1(x), l_2(x)$ and $l_3(x)$.

$$\Rightarrow l_2(x) = \frac{(x-x_0)}{(x_2-x_0)} \frac{(x-x_1)}{(x_2-x_1)} \frac{(x-x_3)}{(x_2-x_3)}$$

$$= \frac{(x-1)}{2} \frac{(x-2)}{1} \frac{(x-4)}{-1}$$

$$= -\frac{1}{2}(x-1)(x-2)(x-4)$$

$$\Rightarrow l_3(x) = \frac{(x-x_0)}{(x_3-x_0)} \frac{(x-x_1)}{(x_3-x_1)} \frac{(x-x_2)}{(x_3-x_2)}$$

$$= \frac{(x-1)}{3} \frac{(x-2)}{2} \frac{(x-3)}{1} = \frac{1}{6}(x-1)(x-2)(x-3)$$

(12)

$$\Rightarrow P_3(x) = 0 \cdot l_0(x) + 2\, l_1(x) + 1\, l_2(x) + 5\, l_3(x)$$

$$\Rightarrow P_3\left(x = \frac{3}{2}\right) = 2 \frac{1}{2}\left(\frac{3}{2} - 1\right)\left(\frac{3}{2} - 3\right)\left(\frac{3}{2} - 4\right)$$

$$+ \left(-\frac{1}{2}\right)\left(\frac{3}{2} - 1\right)\left(\frac{3}{2} - 2\right)\left(\frac{3}{2} - 4\right)$$

$$+ \frac{5}{6}\left(\frac{3}{2} - 1\right)\left(\frac{3}{2} - 2\right)\left(\frac{3}{2} - 3\right)$$

$$= 2 \frac{1}{2} \frac{1}{2}\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)$$

$$+ \left(-\frac{1}{2}\right) \frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{5}{2}\right)$$

$$+ \frac{5}{6} \frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)$$

$$= \frac{1}{16}(30 - 5 + 5) = \frac{15}{8} = 1,875$$

1 Pt. for correct value.

Σ 8

4. $I = \int \sin(x)\,dx = 2$

i) $a = 0,\ b = \pi;\ h = \frac{\pi}{3}$ ① Pt. for the coefficients.

$$I_1 = \frac{3h}{8}\left(\sin(a) + 3\sin(a+h) + 3\sin(a+2h) + \sin(b)\right)$$

$$= \frac{\pi}{8}\left(\sin(0) + 3\sin\left(\frac{\pi}{3}\right) + 3\sin\left(\frac{2\pi}{3}\right) + \sin(\pi)\right)$$

$$= \frac{3\pi}{8}\left(\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2}\right) = \frac{3\sqrt{3}}{8}\pi \approx 2,0405$$

① Pt. for the correc result.

ii) $$I_2 = \underbrace{\int_0^{\pi/2} \sin(x)\,dx}_{=: I_2^{(1)}} + \underbrace{\int_{\pi/2}^{\pi} \sin(x)\,dx}_{=: I_2^{(2)}}$$

① Pt. for the splitting

$I_2^{(1)}$: $a = 0,\ b = \frac{\pi}{2},\ h = \frac{\pi}{6}$

$$\Longrightarrow I_2^{(1)} = \frac{3h}{8}\left(\sin(a) + 3\sin(a+h) + 3\sin(a+2h) + \sin(b)\right)$$

$$= \frac{\pi}{16}\left(\sin(0) + 3\sin\left(\frac{\pi}{6}\right) + 3\sin\left(\frac{\pi}{3}\right) + \sin\left(\frac{\pi}{2}\right)\right)$$

$$= \frac{\pi}{16}\left(\frac{3}{2} + 3\frac{\sqrt{3}}{2} + 1\right)$$

$$= \frac{\pi}{32}\left(5 + 3\sqrt{3}\right)$$

① Pt. for correct result for $I_2^{(1)}$.

$I_2^{(2)}$: $a = \frac{\pi}{2},\ b = \pi,\ h = \frac{\pi}{6}$

$$\Longrightarrow I_2^{(2)} = \frac{3h}{8}\left(\sin(a) + 3\sin(a+h) + 3\sin(a+2h) + \sin(b)\right)$$

$$= \frac{\pi}{16}\left(\sin\left(\frac{\pi}{2}\right) + 3\sin\left(\frac{4\pi}{6}\right) + 3\sin\left(\frac{5\pi}{6}\right) + \sin(\pi)\right)$$

$$= \frac{\pi}{16}\left(1 + 3\frac{\sqrt{3}}{2} + 3\,\frac{1}{2} + 0\right)$$

$$= \frac{\pi}{32}\left(5 + 3\sqrt{3}\right)$$ 1 Pt. for correct result fo $I_2^{(1)}$.

$$\Rightarrow \quad I_2 = I_2^{(1)} + I_2^{(2)} = \frac{\pi}{16}\left(5+3\sqrt{3}\right) \approx 2{,}002009$$ 1 Pt. for the correct result.

iii)

$$I_1 = I + c\,h_1^4 + O(h_1^8)$$

$$I_2 = I + c\,h_2^4 + O(h_2^8)$$

$$= I + c\left(\frac{h_1}{2}\right)^4 + O(h_1^4)$$

$$= I + \frac{c}{16}h_1^4 + O(h_1^4)$$

1 Pt. for the correct error expansions.

$$\Rightarrow \quad I_1 - 16 I_2 = -15 I + c\left(h_1^4 - 16\frac{h_1^4}{16}\right) + O(h_1^8)$$

$$= -15 I + O(h_1^8)$$

$$\Rightarrow \quad I = \frac{I_1 - 16 I_2}{-15} + O(h_1^8)$$

$$\approx 1{,}999441$$

1 Pt. for the correct result.

$\Sigma 8$

5. ①

$$\alpha^4 - 4\alpha^3 + 6\alpha^2 - \frac{9}{4} = 0$$

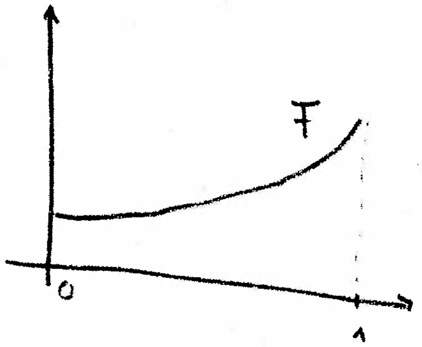
$$\Updownarrow$$

$$6\alpha^2 = \frac{9}{4} - \alpha^4 + 4\alpha^3$$

$$\Updownarrow$$

$$\alpha = \underbrace{\sqrt{\frac{1}{6}\left(\frac{9}{4} - \alpha^4 + 4\alpha^3\right)}}_{=: F(\alpha)}$$

a)



~~(1 Pt for the plot and the statement that F is increasing on (0,1).)~~

$\Rightarrow$ $F(\alpha)$ is increasing thus we only need to check

$$F(0) = \sqrt{\frac{1}{6}\left(\frac{9}{4}\right)} = \sqrt{\frac{9}{24}} \in (0,1)$$

$$F(1) = \sqrt{\frac{1}{6}\left(\frac{9}{4} - 1 + 4\right)} = \sqrt{\frac{1}{6}\,\frac{9-4+16}{4}}$$

$$= \sqrt{\frac{21}{24}} \in (0,1)$$

$\Rightarrow$ $F : [0,1] \longrightarrow [0,1]$

① Pt for the boundary values and conclusion that $F: (0,1) \rightarrow (0,1)$.

Furthermore we need that

$$\exists L < 1 : \quad |F(x) - F(y)| \leq L|x-y| \quad , \quad \forall x,y \in (0,1)$$

$$\Rightarrow \quad |F(x) - F(y)| \leq \|F'\|_\infty |x-y| \quad \forall x,y \in (0,1)$$

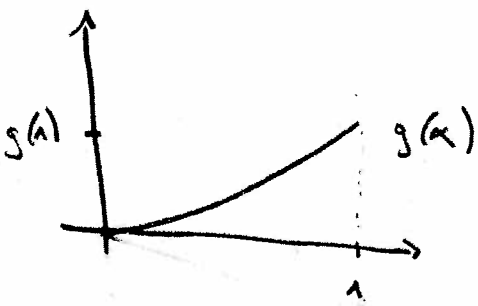
$$\Rightarrow \quad \|F'\|_\infty = \sup_{\alpha \in (0,1)} |F'(\alpha)|$$

$$= \sup_{\alpha \in (0,1)} \left| \frac{1}{2} \left(\frac{1}{6} \left(\frac{9}{4} - \alpha^4 + 4\alpha^3 \right) \right)^{-1/2} (-4\alpha^3 + 12\alpha^2) \right|$$

$$= \sup_{\alpha \in (0,1)} \underbrace{\alpha^2 \frac{\left|1 - \frac{\alpha}{3}\right|}{\sqrt{\frac{1}{6}\left(\frac{9}{4} - \alpha^4 + 4\alpha^3\right)}}}_{=g(\alpha)}$$

(1) Pt for the correct derivative.

By plotting $g(\alpha)$ we get



$g(0) = 0$

$g(1) = 0{,}7126$

(1) Pt for ~~the plot~~ the upper bound on g

Thus $g(\alpha) \leq 0{,}7126 < 1 \ \forall \alpha \in (0,1)$ and thus $L \leq 0{,}7126 < 1$.

$$\rightarrow \quad \alpha_1 = F(\alpha_0) = \sqrt{\frac{1}{6}\left(\frac{9}{4} - \left(\frac{1}{2}\right)^4 + 4\left(\frac{1}{2}\right)^3\right)} = 0{,}6693$$

(1) Pt for correct result

b) i) $f(x) = x^4 - 4x^3 + 6x^2 - \frac{9}{4}$

$$\Rightarrow f'(x) = 4x^3 - 12x^2 + 12x$$ (1) Pt. for the correct derivative

$$\Rightarrow x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{x_n^4 - 4x_n^3 + 6x_n^2 - \frac{9}{4}}{4x_n^3 - 12x_n^2 + 12x_n}$$

(1) Pt. for the correct Newton-Algorithm.

E:)


```
function alphanew=question5(TOL,alpha0)
format long;
alphanew=alpha0;
alphaold=alphanew+1;


while(abs(alphaold-alphanew)>TOL)
    alphaold=alphanew;
    falpha=alphanew^4-4*alphanew^3+6*alphanew^2-(9/4);
    fdalpha=4*alphanew^3-12*alphanew^2+12*alphanew;
    if(abs(fdalpha)< 1e-10)
        disp(['Warning denominator very small stop']);
        return
    end
    alphanew=alphanew-(falpha/fdalpha);
    disp(['The new solution is ', num2str(alphanew)])
end

disp(['The Converged Solution is ',num2str(alphanew)])

%MATLAB output.
%>> question5(1e-5,0.5)
%The new solution is 0.83929
%The new solution is 0.81222
%The new solution is 0.81219
%The new solution is 0.81219
%The Converged Solution is 0.81219
%
%ans =
%
%   0.81218895406217
```

2 Pts for correct MATLAB program.

$\Sigma 8$

6

$$\dot{x}_1 = e^{x_1 x_2} + x_2 - 1.8$$

$$\dot{x}_2 = x_1^2 + x_2^2 - x_1 - 0.8$$

a) impl. Euler: $y_{n+1} = y_n + h f(t_{n+1}, y_{n+1})$ 1P

$$\Leftrightarrow \begin{cases} x_1^{(n+1)} = x_1^{(n)} + h\left(e^{x_1^{(n+1)} x_2^{(n+1)}} + x_2^{(n+1)} - 1.8\right) \\ x_2^{(n+1)} = x_2^{(n)} + h\left((x_1^{(n+1)})^2 + (x_2^{(n+1)})^2 - x_1^{(n+1)} - 0.8\right) \end{cases}$$ 1P

2P

b) $x_1 \hat{=} x_1^{(1)}$, $x_2 \hat{=} x_2^{(1)}$

$$\begin{cases} x_1 = 0.1\left(e^{x_1 x_2} + x_2 - 1.8\right) \\ x_2 = 0.5 + 0.1\left(x_1^2 + x_2^2 - x_1 - 0.8\right) \end{cases}$$

1P Euler-Schritt

$$\Leftrightarrow \begin{cases} x_1 - 0.1\left(e^{x_1 x_2} + x_2 - 1.8\right) = 0 \\ x_2 - 0.1\left(x_1^2 + x_2^2 - x_1 - 0.8\right) \overset{-0.5}{=} 0 \end{cases}$$

1P für Gl.

Quasi-Newton $x_1^{(0)} = 0$, $x_2^{(0)} = 0.5$

1

$$J^{(0)} = \begin{pmatrix} 1 - 0.1\left(x_2^{(0)} e^{x_1^{(0)} x_2^{(0)}}\right) & -0.1\left(x_1^{(0)} e^{x_1^{(0)} x_2^{(0)}} + 1\right) \\ -0.1\left(2x_1^{(0)} - 1\right) & 1 - 0.1\left(2x_2^{(0)}\right) \end{pmatrix}$$

$$= \begin{pmatrix} 1 - 0.1 \cdot 0.5 & -0.1 \\ -0.1(-1) & 1 - 0.1(1) \end{pmatrix} = \begin{pmatrix} 0.95 & -0.1 \\ 0.1 & 0.9 \end{pmatrix}$$ 2P

2) $J^{(0)}\delta = -f(x^{(0)})$

$$\begin{pmatrix} 0.95 & -0.1 \\ 0.1 & 0.9 \end{pmatrix} \begin{pmatrix} \delta_1 \\ \delta_2 \end{pmatrix} = \begin{pmatrix} 0.1(1+0.5-1.8) \\ -0.5+0.1(0.5^2-0.8)+0.5 \end{pmatrix} = \begin{pmatrix} -0.03 \\ -0.055 \end{pmatrix}$$

$$\Leftrightarrow \underbrace{\begin{pmatrix} 1 & 0 \\ 0{,}1053 & 1 \end{pmatrix}}_{L} \underbrace{\begin{pmatrix} 0.95 & -0.1 \\ 0 & 0.9105 \end{pmatrix}}_{R} \begin{pmatrix} \delta_1 \\ \delta_2 \end{pmatrix} = \begin{pmatrix} -0.03 \\ -0.055 \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} \delta_1 \\ \delta_2 \end{pmatrix} = \cancel{\begin{pmatrix} -0.0249 \\ 0.6639 \end{pmatrix}} = \begin{pmatrix} -0.0376 \\ -0.0569 \end{pmatrix}$$

3) $x^{(1)} = x^{(0)} + \delta$

$$x_1^{(1)} = 0 + (-0.0249) = \cancel{-0.0249} \quad -0.0376$$

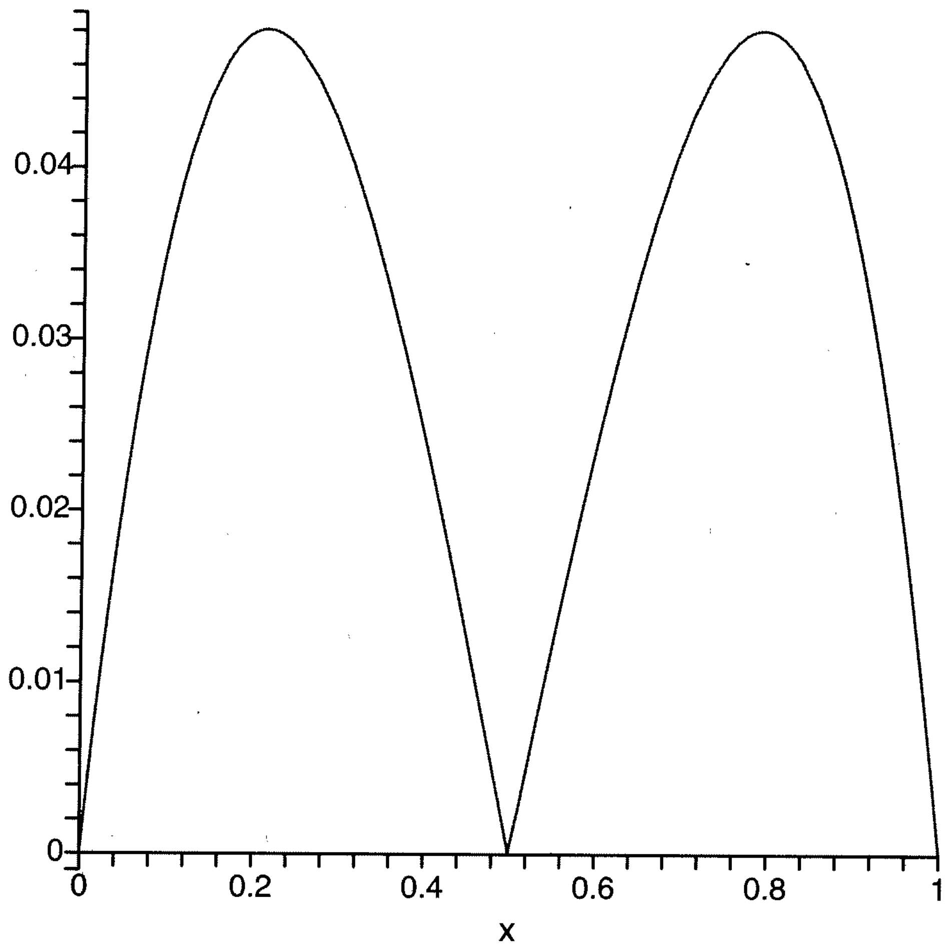
$$x_2^{(1)} = 0.5 + 0.6639 = \cancel{0.5639}\ 0.4431 \quad \text{ZP}$$

```
> restart:
> h := 0.5;
  g := x -> x*(x^2-3*h*x+2*h^2);

  plot(abs(g(x)),x=0..2*h);
```

$$h := 0.5$$

$$g := x \to x\,(x^2 - 3\,h\,x + 2\,h^2)$$




```
> F := alpha -> sqrt(1/6*(9/4-alpha^4+4*alpha^3));
  DF := unapply(diff(F(alpha),alpha),alpha);

  plot({F(alpha),abs(DF(alpha))},alpha=0..1);

  'F(0)' = evalf(F(0));
  'F(1)' = evalf(F(1));
```

6.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} e^{x_1 x_2} + x_2 - 1{,}8 \\ x_1^2 + x_2^2 - x_1 - 0{,}8 \end{bmatrix}}_{=: F(x_1, x_2)}$$

a)

$$x_1(t_n) =: x_1^{(n)}, \quad x_2(t_n) =: x_2^{(n)}$$

$$\Rightarrow \begin{bmatrix} x_1^{(n+1)} \\ x_2^{(n+1)} \end{bmatrix} = \begin{bmatrix} x_1^{(n)} \\ x_2^{(n)} \end{bmatrix} + h F\left(x_1^{(n+1)}, x_2^{(n+1)}\right)$$

① Pt. for the implicit Euler scheme

$$\Rightarrow \begin{cases} x_1^{(n+1)} = x_1^{(n)} + h\left(e^{x_1^{(n+1)} x_2^{(n+1)}} + x_2^{(n+1)} - 1{,}8\right) \\ x_2^{(n+1)} = x_2^{(n)} + h\left(x_1^{(n+1)^2} + x_2^{(n+1)^2} - x_1^{(n+1)} - 0{,}8\right) \end{cases}$$

① Pt. for the correct equations

b)

$$\Rightarrow \begin{cases} x_1^{(n+1)} - h\left(e^{x_1^{(n+1)} x_2^{(n+1)}} + x_2^{(n+1)} - 1{,}8\right) - x_1^{(n)} = 0 \\ x_2^{(n+1)} - h\left(x_1^{(n+1)^2} + x_2^{(n+1)^2} - x_1^{(n+1)} - 0{,}8\right) - x_2^{(n)} = 0 \end{cases}$$

For $n = 0$ we obtain the following non-linear set of equations (dropping superscripts for clarity)

$$\Rightarrow \begin{cases} x_1 - h\left(e^{x_1 x_2} + x_2 - 1{,}8\right) - 0 = 0 \\ x_2 - h\left(x_1^2 + x_2^2 - x_1 - 0{,}8\right) - 0{,}5 = 0 \end{cases}$$

① Pt. for the non-linear system

$$\Rightarrow \begin{cases} x_1 - h\left(e^{x_1 x_2} + x_2 - 1{,}8\right) & = 0 \\ x_2 - h\left(x_1^2 + x_2^2 - x_1 - 0{,}8\right) - 0{,}5 = & 0 \end{cases}$$

$$\underbrace{\qquad\qquad\qquad\qquad}_{=: K(x_1, x_2)}$$

We solve this with the non-linear Quasi-Newton algorithm (introducing subscripts j for Q-N iterations)

- $$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\Bigg|_{j+1} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\Bigg|_{j} + \delta\Big|_j$$

$$J\Big|_{j=0} \; \delta\Big|_j = -K(x_1, x_2)$$

① Pt for the Q-N algorithm for the non-lin[ear] system.

with

- $$J = \begin{bmatrix} \frac{\partial K_1}{\partial x_1} & \frac{\partial K_1}{\partial x_2} \\ \frac{\partial K_2}{\partial x_1} & \frac{\partial K_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 1 - h x_2 e^{x_1 x_2} & h(x_1 e^{x_1 x_2} + 1) \\ -h(2x_1 - 1) & 1 - h\, 2x_2 \end{bmatrix}$$

$$\Rightarrow J\Big|_{j=0} = \begin{bmatrix} 1 - \frac{h}{2} & h \\ h & 1-h \end{bmatrix} = \begin{bmatrix} 0{,}95 & 0{,}1 \\ 0{,}1 & 0{,}9 \end{bmatrix}$$

② Pts. for the correct Jacobi matrix

$$J = \begin{bmatrix} 0{,}95 & 0{,}1 \\ 0{,}1 & 0{,}9 \end{bmatrix}$$

$$\xrightarrow{Z_2 - {}^{0{,}1}/_{0{,}95} Z_1} \begin{bmatrix} 0{,}95 & 0{,}1 \\ 0 & 0{,}8895 \end{bmatrix}$$

$$\Rightarrow \quad L = \begin{bmatrix} 1 & 0 \\ 0{,}1053 & 1 \end{bmatrix}$$

$$\Rightarrow \quad R = \begin{bmatrix} 0{,}95 & 0{,}1 \\ 0 & 0{,}8895 \end{bmatrix}$$

$$\Rightarrow \quad K(0,\ 0{,}5) = \begin{bmatrix} 0 - h\,(1 + 0{,}5 - 1{,}8) \\ 0{,}5 - h\,(0 + (0{,}5)^2 - 0 - 0{,}8) - 0{,}5 \end{bmatrix}$$

$$= h \begin{bmatrix} 0{,}3 \\ 0{,}55 \end{bmatrix}$$

$$= \begin{bmatrix} 0{,}03 \\ 0{,}055 \end{bmatrix}$$

Thus we obtain

$$\vec{s}\,\Big|_{j=0} = \begin{bmatrix} 0{,}0254 \\ 0{,}0583 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1^{(2)} \\ x_2^{(2)} \end{bmatrix} = \begin{bmatrix} 0{,}0254 \\ 0{,}5583 \end{bmatrix}$$

1 Pt. for the correct result.