SS06

1) a) E.W. $|A-\lambda I| = 0 \iff \begin{vmatrix} 4-\lambda & 2 & -2 \\ 2 & 5-\lambda & 0 \\ -2 & 0 & 3-\lambda \end{vmatrix} = 0$

nach 1. Spalte: $(4-\lambda)[(5-\lambda)(3-\lambda)-0] - 2[2(3-\lambda)-0] - 2[2(5-\lambda)] = 0$

$\iff (4-\lambda)(5-\lambda)(3-\lambda) - 4(3-\lambda) - 4(5-\lambda) = 0$

$\iff (20-9\lambda+\lambda^2)(3-\lambda) - 12 + 4\lambda - 20 + 4\lambda = 0$

$\iff 60 - 27\lambda + 3\lambda^2 - 20\lambda + 9\lambda^2 - \lambda^3 - 32 + 8\lambda = 0$

$\iff -\lambda^3 + 12\lambda^2 - 39\lambda + 28 = 0$

$\lambda_1 = 1$ ist Lösung

$\Rightarrow -\lambda^3 + 12\lambda^2 - 39\lambda + 28 : (\lambda - 1) = -\lambda^2 + 11\lambda - 28$

$-\lambda^3 + \lambda^2$

$11\lambda^2$

$11\lambda^2 - 11\lambda$

$28\lambda + 28$

$\Rightarrow (\lambda-1)(-\lambda^2 + 11\lambda - 28) = 0$, $\Delta = 121 - 4(-1)(-28) = 9$

$\Rightarrow \lambda_2 = \frac{-11+3}{-2} = 4$

$\lambda_3 = -\frac{11-3}{2} = 7$

$\Rightarrow$ i) Matrix A ist positiv definit: alle E.W. > 0

ii) Matrix A ist symmetrisch

$\Rightarrow$ CG anwendbar

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b) 1. Schritt: $Au + d = 0$ mit $d = -b = -\begin{pmatrix} 1 \\ 5 \\ 7 \end{pmatrix}$, $u^0 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$r^0 = Au^0 + d = \begin{pmatrix} 4 & 2 & -2 \\ 2 & 5 & 0 \\ -2 & 0 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 5 \\ 7 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ -6 \end{pmatrix}$

$p^1 = -r^0$, $\rho_1 = \frac{(r^0, r^0)}{(p^1, Ap^1)}$, $A \cdot p^1 = \begin{pmatrix} 4 & 2 & -2 \\ 2 & 5 & 0 \\ -2 & 0 & 3 \end{pmatrix}\begin{pmatrix} -3 \\ -2 \\ 6 \end{pmatrix} = \begin{pmatrix} -28 \\ -16 \\ 24 \end{pmatrix}$

$p^1 \cdot (Ap^1) = \begin{pmatrix} -3 \\ -2 \\ 6 \end{pmatrix} \cdot \begin{pmatrix} -28 \\ -16 \\ 24 \end{pmatrix} = 260$

$r^0 \cdot r^0 = \begin{pmatrix} 3 \\ 2 \\ -6 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 2 \\ -6 \end{pmatrix} = 9 + 4 + 36 = 49$

$\Rightarrow \rho^1 = \frac{49}{260}$

$$u^1 = u^0 + \rho_1 p^1 = \begin{pmatrix}1\\1\\1\end{pmatrix} + \frac{49}{260}\begin{pmatrix}-3\\-2\\6\end{pmatrix} = \begin{pmatrix}1\\1\\1\end{pmatrix} + \begin{pmatrix}-147/260\\-98/260\\294/260\end{pmatrix} = \underline{\underline{\frac{1}{260}\cdot\begin{pmatrix}113\\162\\554\end{pmatrix}}} \quad 550$$

c) CG ist exakt nach spätestens n Schritt für $n \times n$ Matrix, also hier <u>$n=3$ Schritten</u>

(2)

a) $x^0 = (3,1,1,2)^T$, Löse $\vec{F}(\vec{x}) = 0$ mit

$$\vec{F}(x_1,x_2,x_3,x_4) = \begin{pmatrix} x_2^2 + x_1 - \alpha \\ x_2^2 + 3 - \beta \\ 4 + x_4^3 - \gamma \\ x_4^3 + x_3 - \delta \end{pmatrix}, \quad \vec{J}(x_1,x_2,x_3,x_4) = \begin{pmatrix} \frac{\partial F_1}{\partial x_1} & \cdots & \frac{\partial F_1}{\partial x_4} \\ \vdots & & \\ \frac{\partial F_4}{\partial x_1} & \cdots & \frac{\partial F_4}{\partial x_4} \end{pmatrix}$$

$$J = \left(\begin{array}{cc|cc} 1 & 2x_2 & 0 & 0 \\ 0 & 2x_2 & 0 & 0 \\ \hline 0 & 0 & 0 & 3x_4^2 \\ 0 & 0 & 1 & 3x_4^2 \end{array}\right) \equiv \left(\begin{array}{c|c} A_1 & 0 \\ \hline 0 & A_2 \end{array}\right); \quad \begin{array}{l} \det(A_1) = -2x_2 \\ \det(A_2) = -3x_4^2 \end{array}$$

$$x^1 = x^0 - J^{-1}\Big|_{x_0} \cdot \vec{F}(x^0) = \begin{pmatrix} 3 \\ 1 \\ 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 1 & 3 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 1 - 3 - \alpha \\ 1 + 3 - \beta \\ 4 + 8 - \gamma \\ 8 + 1 - \delta \end{pmatrix}$$

$$= \begin{pmatrix} 3 \\ 1 \\ 1 \\ 2 \end{pmatrix} - \begin{pmatrix} -1 & -1 & 0 & 0 \\ 0 & -1/2 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1/3 & 0 \end{pmatrix} \begin{pmatrix} -2 - \alpha \\ 4 - \beta \\ 12 - \gamma \\ 9 - \delta \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 2 + \alpha - 4 + \beta \\ -2 + \beta/2 \\ 12 - \gamma - 9 + \delta \\ 4 - \gamma/3 \end{pmatrix}$$

$$\Rightarrow x^1 = \begin{pmatrix} 3 - \alpha - \beta \\ 3 - \beta/2 \\ -1 + \gamma - \delta \\ -2 + \gamma/3 \end{pmatrix}$$

b) $x^\lambda = -e^x + 7$ $\qquad \lambda = 3!$

$\Rightarrow$ i) `function f = newton_verfahren(~~lambda~~, a)`

`f = a^lambda + exp(a)-7;` `% f = x^λ + e^x - 7` $\quad // \; f = x^3 + e^x - 7$

ii) `function f = newton_verfahren_deriv(~~lambda~~, a)`

`f = ~~lambda~~ 3 * a^(~~lambda-1~~ 2) + exp(a);` `% f' = λx^(λ-1) + e^x` $\quad // \; f' = 3x^2 + e^x$

iii)

```
% λ = ... ; gegeben λ=3!
format long;
a=1;   % x^0
b=0;   % initializierung
while abs(a-b) > 10^(-6)
    b=a;
    a = a-feval('newton_verfahren', ~~lambda~~, a)/feval('newton_verfahren_deriv', ~~lambda~~, a)  % x^(k+1) = x^k - f(x^k)/f'(x^k)
end
disp(a);
```

SS0

(3) a) $P_2(x) \to$ 3 Stützpunkten, äquidistant auf $I=[0,2]$

$$\Rightarrow x_0=0,\ f_0=1/2$$
$$x_1=1,\ f_0=1$$
$$x_2=2,\ f_2=1/2$$

$$P_2(x)=\sum_{i=0}^{2} f_i\cdot l_i(x)\ ;\quad l_i(x)=\prod_{j\neq i}^{2}\frac{x-x_j}{x_i-x_j}$$

$$l_0(x)=\frac{x-1}{-1}\cdot\frac{x-2}{-2}=(1-x)\frac{(2-x)}{2}=\frac{2-3x+x^2}{2}$$
$$l_1(x)=\frac{x}{1}\cdot\frac{x-2}{-1}=x\cdot\frac{(x-2)}{-1}=2x-x^2$$
$$l_2(x)=\frac{x}{2}\cdot\frac{x-1}{1}=\frac{x}{2}\cdot\frac{(x-1)}{1}=\frac{x^2-x}{2}$$

$$\Rightarrow P_2(x)=\frac{1}{2}\left(\frac{2-3x+x^2}{2}\right)+2x-x^2+\frac{1}{2}\left(\frac{x^2-x}{2}\right)=\underline{\underline{-\frac{x^2}{2}+x+\frac{1}{2}}}$$

b) $$D'(x)=\frac{-2(x-1)}{[1+(x-1)^2]^2}+x-1\overset{!}{=}0$$

$$x_0=0$$
$$D'(x_0)=\frac{2}{(1+1)^2}-1=-0,5$$
$$x_1=0,5$$
$$D'(x_1)=\frac{-2(-0,5)}{[1+0,25]^2}+0.5-1=0.14$$

$$\to x^{k+1}=\frac{x_{k-1}\cdot f(x_k)-x_k\cdot f(x_{k-1})}{f(x_k)-f(x_{k-1})}$$

$$\Rightarrow x^2=\frac{x_0\cdot D'(x_1)-x_1\cdot D'(x_0)}{D'(x_1)-D'(x_0)}=\frac{0-0.5(-0.5)}{0.14-(-0.5)}$$

$$=\frac{0.25}{0.64}=\underline{\underline{0.3906=x^*}}$$

4

a) $I = \int_0^1 \alpha \cdot f(x)\,dx = \alpha \cdot \int_0^1 f(x)\,dx = \frac{\alpha}{2}\int_{-1}^1 f(x)\,dx = \frac{\alpha}{2}\sum_{i=1}^5 w_i \cdot f(x_i)$

f symmetrisch: $f(x) = f(-x)$

$\Rightarrow I = \frac{\alpha}{2}\left[0.5689 \cdot 1 + 0.4786 \cdot 0.5801 \cdot \underset{\text{symm.}}{2} + 0.2369 \cdot 0.2415 \cdot \underset{\text{symm.}}{2}\right]$

wo $f(0{,}9062) = f(-0{,}9062) = 0{,}2415$

$f(0{,}5385) = f(-0{,}5385) = 0{,}5801$

$f(0) = 1$

$I = \frac{\alpha}{2} \cdot 1{,}2386$, $I \stackrel{!}{=} 1 \Leftrightarrow \frac{\alpha}{2} \cdot 1{,}2386 \stackrel{!}{=} 1 \Rightarrow \alpha = \frac{2}{1{,}2386} = \underline{\underline{1{,}6147}}$

b) $\alpha = 1$, $S(1) = \frac{h}{3}\left(f(0) + 4 \cdot f\left(\frac{1}{2}\right) + f(1)\right)$, mit $h = \frac{1}{2}$

$= \underbrace{\frac{h}{3}}_{\frac{1}{6}}\left(1 + 4 \cdot \frac{e^{-1/4}}{1+1/4} + \frac{e^{-1}}{2}\right) = \underline{\underline{0{,}6127}}$ (exakt 0,6192)

c) `s1 = 1/6 * (feval('~~integral~~Y', alpha, 0) + 4 * feval('~~integral~~Y', alpha, 0,5)`
`+ feval('~~integral~~Y', alpha, 1));`

⑤ Euler-explizit:

a)
$$\begin{pmatrix} x_1^{k+1} \\ x_2^{k+1} \end{pmatrix} = \begin{pmatrix} x_1^{k} \\ x_2^{k} \end{pmatrix} + h \cdot \begin{pmatrix} f_1(x_1^k, x_2^k) \\ f_2(x_1^k, x_2^k) \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_1^{k} \\ x_2^{k} \end{pmatrix} + \underbrace{\begin{pmatrix} 0 & h \\ -h & 0 \end{pmatrix}}_{A = h \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}\begin{pmatrix} x_1^{k} \\ x_2^{k} \end{pmatrix} \overset{!}{=} \overbrace{\begin{pmatrix} 1 & h \\ -h & 1 \end{pmatrix}}^{\text{gegeben}}\begin{pmatrix} x_1^{k} \\ x_2^{k} \end{pmatrix}$$

$$\Rightarrow \frac{d}{dt}\begin{pmatrix} x_1^{k} \\ x_2^{k} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x_1^{k} \\ x_2^{k} \end{pmatrix} \Rightarrow \begin{cases} f_1 = x_2 \\ f_2 = -x_1 \end{cases} \Rightarrow \frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ -x_2 \end{pmatrix} = f$$

b) Implizit-Euler:

$$x^{k+1} = x^k + h \cdot x^{k+1}$$

$$\Rightarrow x^{k+1} - h \cdot x^{k+1} = x^k$$

$$(I - A \cdot h)\, x^{k+1} = x^k$$

$$x^{k+1} = (I - A \cdot h)^{-1} \cdot x^k = \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & h \\ -h & 0 \end{pmatrix} \right)^{-1} \cdot x^k$$

$$\Rightarrow x^{k+1} = \begin{pmatrix} 1 & -h \\ h & 1 \end{pmatrix}^{-1} \cdot \cancel{A}x^k = \underbrace{\frac{1}{1+h^2}\begin{pmatrix} 1 & h \\ -h & 1 \end{pmatrix}}_{B(h)} \cdot x^k$$

c)
$$\|x^{k+1}\| = \|A \cdot x^k\| \leq \|A\| \, \|x^k\|$$

mit $\|A\|_2 = \sqrt{\max(\operatorname{eig}(A^T A))}$, $A^T A = \begin{pmatrix} 1+h^2 & 0 \\ 0 & 1+h^2 \end{pmatrix} \cdot \left(\frac{1}{1+h^2}\right)^2$

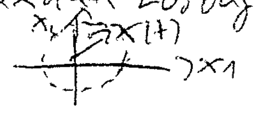
$$A^T A = \begin{pmatrix} \frac{1}{1+h^2} & 0 \\ 0 & \frac{1}{1+h^2} \end{pmatrix}$$

$$\Rightarrow \|A\|_2 = \sqrt{\frac{1}{1+h^2}} < 1$$

$$\Rightarrow \|x^{j+1}\| \leq \|A\| \cdot \|x^j\| = \underbrace{\sqrt{\frac{1}{1+h^2}}}_{<1} \cdot \|x^0\| \Rightarrow \lim_{j \to \infty} \|x^j\| = 0$$

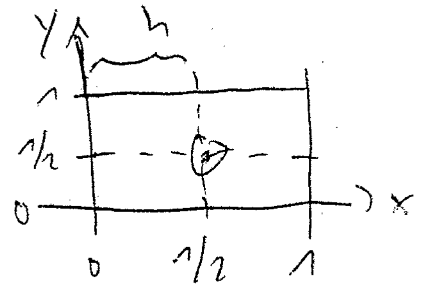
d) Diese Gleichung ist die harmonische oscillator, exakte Lösung $\equiv$ Kreis

$\|x(t)\| = \text{const}$, rotiert in (x_1, x_2) ebene



(6)

a) Gitter



nur 1 innere Punkte am $\left(\frac{1}{2}, \frac{1}{2}\right)$

$$\Rightarrow -\Delta u\left(\tfrac{1}{2}, \tfrac{1}{2}\right) = -\frac{1}{h^2}\left[u\left(0, \tfrac{1}{2}\right) + u\left(\tfrac{1}{2}, 1\right) + u\left(1, \tfrac{1}{2}\right) + u\left(\tfrac{1}{2}, 0\right) - 4 \cdot u\left(\tfrac{1}{2}, \tfrac{1}{2}\right)\right]$$

Randbedingung: $u(x, y) = x \Rightarrow u\left(0, \frac{1}{2}\right) = 0$

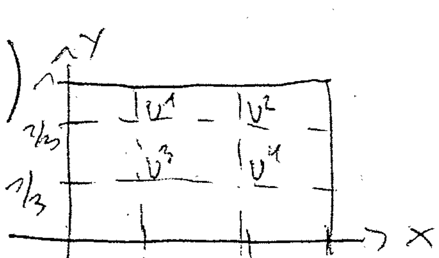
$$u\left(\tfrac{1}{2}, 1\right) = 1/2$$
$$u\left(1, 1/2\right) = 1$$
$$u\left(\tfrac{1}{2}, 0\right) = \tfrac{1}{2}$$

$\Rightarrow$ im obere Formula einzetzen:

$$-\Delta u\left(\tfrac{1}{2}, \tfrac{1}{2}\right) = -\frac{1}{h^2}\left(0 + \tfrac{1}{2} + 1 + \tfrac{1}{2} - 4 \cdot u\left(\tfrac{1}{2}, \tfrac{1}{2}\right)\right)$$
$$= -\frac{1}{h^2}\left(2 - 4u\left(\tfrac{1}{2}, \tfrac{1}{2}\right)\right) = 1$$

$$\Rightarrow 2 - 4u\left(\tfrac{1}{2}, \tfrac{1}{2}\right) = -h^2$$
$$-4 \cdot u\left(\tfrac{1}{2}, \tfrac{1}{2}\right) = -h^2 - 2 \Rightarrow u\left(\tfrac{1}{2}, \tfrac{1}{2}\right) = \frac{h^2 + 2}{4} = \frac{1}{16} + \frac{2}{4} = \underline{\underline{\frac{9}{16}}}$$

b)



$\Rightarrow$ 4 innere Punkte

Randbedingung: $u(x, y) \equiv 0$ auf $\partial\Omega$

$$\begin{cases} -\Delta v^1 = -\frac{1}{h^2}\left[v^3 + v^2 - 4v^1\right] \\ -\Delta v^2 = -\frac{1}{h^2}\left[v^1 + v^4 - 4v^2\right] \\ -\Delta v^3 = -\frac{1}{h^2}\left[v^1 + v^4 - 3v^3\right] \\ -\Delta v^4 = -\frac{1}{h^2}\left[v^2 + v^3 - 4v^4\right] \end{cases}$$

$$\Rightarrow -\Delta \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} = -\frac{1}{h^2}\begin{bmatrix} -4 & 1 & 1 & 0 \\ 1 & -4 & 0 & 1 \\ 1 & 0 & -4 & 1 \\ 0 & 1 & 1 & -4 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

mit $h = \frac{1}{3}$

$$\Rightarrow -9 \cdot \begin{bmatrix} -4 & 1 & 1 & 0 \\ 1 & -4 & 0 & 1 \\ 1 & 0 & -4 & 1 \\ 0 & 1 & 1 & -4 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$\Rightarrow$ Lösung: $\underline{v_1 = v_2 = v_3 = v_4 =}$ $+0{,}05\bar{5} =$ [illegible]