ETHZ, BSc D-MAVT
Prüfung Herbst 07
Numerische Mathematik Lösungen
Prof. K. Nipp

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1. a)

$$T = D^{-1}(-L-R) = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/3 \end{pmatrix} \begin{pmatrix} 0 & -\alpha \\ -\beta & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\frac{\alpha}{2} \\ -\frac{\beta}{3} & 0 \end{pmatrix} \quad c = (\frac{1}{2}, \frac{2}{3})'$$

b)

$$T = \begin{pmatrix} 0 & -\frac{1}{2} \\ -\frac{2}{3} & 0 \end{pmatrix}$$

mit $det(\lambda Id - T) = \lambda^2 - \frac{1}{3} = 0 \;\Rightarrow\; \lambda_{1,2} = \pm\sqrt{\frac{1}{3}}$ Also ist der Spektralradius $\varrho(T) \approx 0.5774 < 1$ und das Verfahren konvergiert.

c)

$$||T||_2 = \sqrt{\max|eig(T' * T)|}$$

$$T' * T = \begin{pmatrix} \frac{4}{9} & 0 \\ 0 & \frac{1}{4} \end{pmatrix}$$

Also $||T||_2 = \frac{2}{3}$. Gesucht ist nun n sodass $||T||_2^n \leq 0.1$

$$\left(\frac{2}{3}\right)^n \leq \frac{1}{10}$$

$$n\log(\frac{2}{3}) \leq \log(\frac{1}{10})$$

$$n \geq \frac{\log(1/10)}{\log(2/3)} = 5.67$$

$$\Rightarrow n = 6$$

d) $\alpha = \beta$ (A symmetrisch) Eigenwerte müssen grösser 0 sein

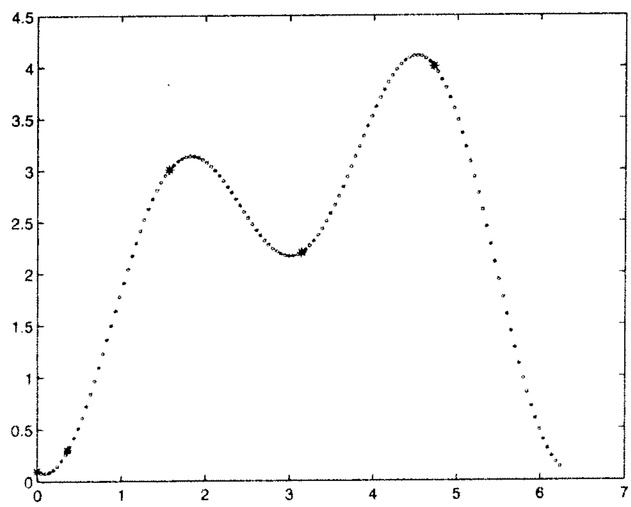
$$det(\lambda Id - A) = (\lambda-2)(\lambda-3) - \alpha^2 = \lambda^2 - 5\lambda + 6 - \alpha^2 = 0$$

$$\lambda_{1,2} = \frac{5 \pm \sqrt{25 - 24 + 4\alpha^2}}{2} > 0$$

$$1 + 4\alpha^2 < 25$$

$$\alpha \in (-\sqrt{6}, \sqrt{6})$$

2.
```
close all;
clear all;

f=[0.1 3 2.2 4];

N=length(f);
m=5;
M=N*2^m;

x=2*pi/N*[0:N-1];

% Plotten der discreten Funktionswerte
plot(x,f,'*');
hold on
%Berechnen der komplexen Koeffizienten
c=fft(f/N);
%Einfuegen von M-N Nullen
c_star=[c(1:N/2),zeros(1,M-N),c(N/2+1:end)];
%Auswertung des trigonometrischen Polynoms auf feinem Gitter
x_star=2*pi/M*[0:M-1];
p=M*ifft(c_star);

plot(x_star,p,'go','MarkerSize',2);

x_=0.37
idx=floor(x_/(2*pi)*M)+1;
p_low=p(idx);
p_up=p(idx+1)
x_low=x_star(idx);
dx=x_-x_low;
h=2*pi/M;
value=real(p_low+dx/h*(p_up-p_low))
%plot(x_,value,'r*')
```



3. **a)**

$$f(x,y) := \frac{xcos(x+y)}{\pi}$$

$$\begin{aligned}
I_x(0) &\approx h\left(\frac{1}{2}f(0,0) + f(\pi/2,0) + \frac{1}{2}f(\pi,0)\right) = \frac{\pi}{2}\left(\frac{1}{2}\cdot 0 + 0 + \frac{1}{2}(-1)\right) = -\frac{\pi}{4} \\
I_x(\pi/2) &\approx h\left(\frac{1}{2}f(0,\pi/2) + f(\pi/2,\pi/2) + \frac{1}{2}f(\pi,\pi/2)\right) = \frac{\pi}{2}\left(\frac{1}{2}\cdot 0 - \frac{1}{2} + \frac{1}{2}\cdot 0\right) = -\frac{\pi}{4} \\
I_x(\pi) &\approx h\left(\frac{1}{2}f(0,\pi) + f(\pi/2,\pi) + \frac{1}{2}f(\pi,\pi)\right) = \frac{\pi}{2}\left(\frac{1}{2}\cdot 0 + 0 + \frac{1}{2}\cdot 1\right) = \frac{\pi}{4}
\end{aligned}$$

b)

$$\begin{aligned}
I &\approx h\left(\frac{1}{2}I_x(0) + I_x(\pi/2) + \frac{1}{2}I_x(\pi)\right) \\
&\approx \frac{\pi}{2}\left(-\frac{\pi}{8} - \frac{\pi}{4} + \frac{\pi}{8}\right) = -\frac{\pi^2}{8}
\end{aligned}$$

4. **a)**

$$l_0(y) = \frac{(y+1)(y-2)}{4}, \quad l_1(y) = \frac{(y+2)(y-2)}{-3}, \quad l_2(y) = \frac{(y+2)(y+1)}{12}$$

$$x(y) = 0\cdot l_0(y) + l_1(y) + 3l_2(y) = \frac{(y+2)(y-2)}{-3} + \frac{(y+2)(y+1)}{4}$$

b)

$$x(0) = \frac{4}{3} + \frac{1}{2} = \frac{8+3}{6} = \frac{11}{6}$$

5. a)

$$\begin{aligned} F(x,t,h) &= F(x,h) = x + f(x)h + \frac{h^2}{2} f_x(x) f(x) \\ &= x + h(\sin x + \frac{1}{2}) + \frac{h^2}{2}\cos x(\sin x + \frac{1}{2}) \end{aligned}$$

b)

$$\begin{aligned} \tilde{x}_1 &= \tilde{x}_0 + h(\sin \tilde{x}_0 + \frac{1}{2}) + \frac{h^2}{2}\cos \tilde{x}_0(\sin \tilde{x}_0 + \frac{1}{2}) \\ &= \frac{\pi}{2} + \frac{1}{50}\left(1 + \frac{1}{2}\right) + 0 = \frac{\pi + 3/50}{\pi} \approx 1.600796326794897 \end{aligned}$$

c)

$$\begin{aligned} \hat{x}_1 &= \tilde{x}_0 + h(\sin \tilde{x}_0 + \frac{1}{2}) + \frac{h^2}{2}\cos \tilde{x}_0(\sin \tilde{x}_0 + \frac{1}{2}) \\ &= \frac{\pi}{2} + \frac{1}{100}\left(1 + \frac{1}{2}\right) + 0 = \frac{\pi + 3/100}{2} \approx 1.585796326794896 \\ \hat{x}_2 &= \hat{x}_1 + h(\sin \hat{x}_1 + \frac{1}{2}) + \frac{h^2}{2}\cos \hat{x}_1(\sin \hat{x}_1 + \frac{1}{2}) \\ &\approx 1.600794076942547 \\ \Delta &= \tilde{x}_1 - \hat{x}_2 = 1.600796326794897 - 1.600794076942547 \approx -2.24985e-06 \end{aligned}$$

$p = 2$ Also $|x - \hat{x}_2| = \frac{\Delta}{2^2-1} \approx 2.24985e - 06/3 \approx 7.4995e - 07$

Alternativ :

$$x = \frac{\tilde{x}_1 - 4\hat{x}_2}{1-4} = \frac{4\hat{x}_2 - \tilde{x}_1}{3} \approx 1.60079332699176 \Rightarrow |x - \hat{x}_2| \approx 7.4995e - 07$$

$$x = \frac{\hat{x}_2 - \frac{1}{4}\tilde{x}_1}{1 - \frac{1}{4}} = \frac{4\hat{x}_2 - \tilde{x}_1}{3} \approx 1.60079332699176 \Rightarrow |x - \hat{x}_2| \approx 7.4995e - 07$$

6. $h = 2/3,\ a = 0.5$

Gleichung fuer $u_1 = u(x_1, y_1)$

$$\begin{aligned} u_{xx} &\approx \frac{2}{h^2}\left(\frac{u_2}{1+a} + \frac{1}{a(1+a)} - \frac{1}{a}u_1\right) = \frac{9}{2}\left(\frac{2}{3}u_2 + \frac{4}{3} - 2u_1\right) \\ &= 3u_2 + 6 - 9u_1 = -9u_1 + 3u_2 + 6 \\ u_{yy} &\approx \frac{1}{(h/2)^2}(1 + 1 - 2u_1) = 9(-2u_1 + 2) = -18u_1 + 18 \end{aligned}$$

$$\begin{aligned} -\Delta u &\approx 27u_1 - 3u_2 - 24 \\ \Rightarrow 3(27u_1 - 3u_2 - 24) &= 2 \\ \Rightarrow 9(9u_1 - u_2) &= 74 \end{aligned}$$

Gleichung fuer $u_2 = u(x_2, y_2)$

$$\Rightarrow 9(9u_1 - u_2) = 76$$

$$9\begin{pmatrix} 9 & -1 \\ -1 & 9 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 74 \\ 76 \end{pmatrix}$$