

# 6

# Polynomial Interpolation

Throughout this chapter  $n \in \mathbb{N}_0$  is an element from the set  $\mathbb{N}_0$  if not otherwise specified.

## 6.1 Polynomials

**Definition 6.1.1.** We denote by

$$\mathcal{P}_n := \left\{ \mathbb{R} \ni t \mapsto \alpha_n t^n + \alpha_{n-1} t^{n-1} + \dots + \alpha_1 t + \alpha_0 \in \mathbb{R} : \alpha_0, \alpha_1, \dots, \alpha_n \in \mathbb{R} \right\} .$$

the  $\mathbb{R}$ -vector space of polynomials with degree  $\leq n$ .

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**Theorem 6.1.2** (Dimension of space of polynomials). It holds that  $\dim \mathcal{P}_n = n + 1$  and  $\mathcal{P}_n \subset C^\infty(\mathbb{R})$ .

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Remark 6.1.2 (Horner scheme). Evaluation of a polynomial in monomial representation: Horner scheme

$$p(t) = t(t \cdots t(t(\alpha_n t + \alpha_{n-1}) + \alpha_{n-2}) + \cdots + \alpha_1) + \alpha_0 .$$

Code 6.1: Horner scheme, polynomial in MATLAB format

```
function y = polyval(p,x)
y = p(1); for i=2:length(p), y = x*y+p(i); end
```

Asymptotic complexity:  $O(n)$

Use: MATLAB “built-in”-function `polyval(p,x);`



## 6.2 Polynomial Interpolation: Theory

Goal:

(re-)construction of a polynomial (function) from pairs of values (fit).

*Lagrange polynomial interpolation problem*

Given the **nodes**  $-\infty < t_0 < t_1 < \dots < t_n < \infty$  and the values  $y_0, \dots, y_n \in \mathbb{R}$  compute  $p \in \mathcal{P}_n$  such that

$$\forall j \in \{0, 1, \dots, n\}: \quad p(t_j) = y_j .$$

## 6.2.1 Lagrange polynomials

**Definition 6.2.1.** Consider *nodes*  $-\infty < t_0 < t_1 < \dots < t_n < \infty$  ( $\rightarrow$  Lagrange interpolation). Then define the *Lagrange polynomials*  $L_0, L_1, \dots, L_n \in \mathcal{P}_n$  associated to  $(t_0, \dots, t_n)$  through

$$L_i(t) := \prod_{\substack{j=0 \\ j \neq i}}^n \frac{(t - t_j)}{(t_i - t_j)}$$

for all  $t \in \mathbb{R}$  and all  $i \in \{0, 1, \dots, n\}$ .

Note that  $L_i(t_j) = \delta_{i,j} = \begin{cases} 1 & : i = j \\ 0 & : i \neq j \end{cases} .$

*Example 6.2.1.* Consider  $n \in \mathbb{N}$  and Lagrange polynomials for uniformly spaced nodes

$$t_j = -1 + \frac{2j}{n}, j \in \{0, 1, \dots, n\}.$$

Plot  $n = 10$ ,  $j = 0, 2, 5 \rightarrow$

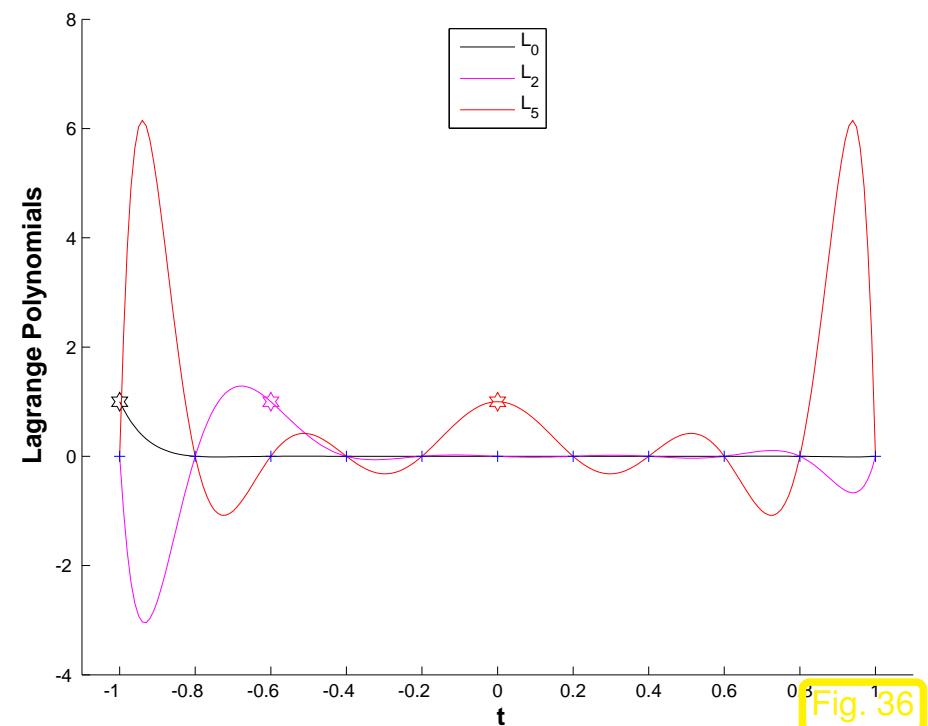


Fig. 36

**Definition 6.2.2** (Lagrange interpolation operator). Consider  $-\infty < t_0 < t_1 < \dots < t_n < \infty$ . By  $\mathbf{I}_{t_0, \dots, t_n}: \mathbb{R}^{n+1} \rightarrow \mathcal{P}_n$  we denote the linear mapping defined by

$$\mathbb{R}^{n+1} \ni (y_0, \dots, y_n) \mapsto \mathbf{I}_{t_0, \dots, t_n}(y_0, \dots, y_n) := \sum_{i=0}^n y_i L_i \in \mathcal{P}_n.$$

**Theorem 6.2.3** (Existence & uniqueness of Lagrange interpolation polynomial).

*The general polynomial interpolation problem (6.2) admits a unique solution  $p \in \mathcal{P}_n$  and it holds that*

$$p(t) = \sum_{i=0}^n y_i \cdot L_i(t) = \left( l_{t_0, \dots, t_n}(y_0, \dots, y_n) \right)(t)$$

for all  $t \in \mathbb{R}$ .

## 6.2.2 Conditioning of polynomial interpolation

Necessary for studying the conditioning: norms on vector space of continuous functions  $C(I, \mathbb{R})$  where  $I \subset \mathbb{R}$  is a non-empty bounded and closed interval:

$$\text{supremum norm} \quad \|f\|_{\infty, I} := \sup\{|f(t)| : t \in I\} , \quad (6.2.4)$$

$$L^2\text{-norm} \quad \|f\|_{2, I} := \sqrt{\int_I |f(t)|^2 dt} , \quad (6.2.5)$$

$$L^1\text{-norm} \quad \|f\|_{1, I} := \int_I |f(t)| dt . \quad (6.2.6)$$

**Lemma 6.2.6** (Absolute conditioning of polynomial interpolation). Consider interval  $I \subset \mathbb{R}$ , nodes  $-\infty < t_0 < \dots < t_n < \infty$  and associated Lagrange polynomials  $L_0, \dots, L_n$ . Then the Lagrange interpolation operator satisfy

$$\|I_{t_0, \dots, t_n}\|_{L(\|\cdot\|_\infty, \|\cdot\|_{\infty, I})} = \sup_{\mathbf{y} \in \mathbb{R}^{n+1} \setminus \{0\}} \frac{\|I_{t_0, \dots, t_n}(\mathbf{y})\|_{\infty, I}}{\|\mathbf{y}\|_\infty} = \left\| \sum_{i=0}^n |L_i| \right\|_{\infty, I}, \quad (6.2.7)$$

$$\|I_{t_0, \dots, t_n}\|_{L(\|\cdot\|_2, \|\cdot\|_{2, I})} = \sup_{\mathbf{y} \in \mathbb{R}^{n+1} \setminus \{0\}} \frac{\|I_{t_0, \dots, t_n}(\mathbf{y})\|_{2, I}}{\|\mathbf{y}\|_2} \leq \left( \sum_{i=0}^n \|L_i\|_{2, I}^2 \right)^{\frac{1}{2}}. \quad (6.2.8)$$

Terminology: **Lebesgue constant** of  $t_0, \dots, t_n$  and  $I \subset \mathbb{R}$ :  $\lambda_{t_0, \dots, t_n}(I) := \left\| \sum_{i=0}^n |L_i| \right\|_{\infty, I}$ .

*Example 6.2.4* (Estimation of the Lebesgue constant). Consider  $n \in \mathbb{N}$ ,  $I = [-1, 1]$  and  $t_k =$

$-1 + \frac{2k}{n}, k \in \{0, 1, \dots, n\}$  (uniformly spaced nodes). Then

$$\begin{aligned}\lambda_{t_0, \dots, t_n}(I) &\geq |L_{n/2}(1 - \frac{1}{n})| = \frac{\frac{1}{n} \cdot \frac{1}{n} \cdot \frac{3}{n} \cdot \dots \cdot \frac{n-3}{n} \cdot \frac{n+1}{n} \cdot \dots \cdot \frac{2n-1}{n}}{\left(\frac{2}{n} \cdot \frac{4}{n} \cdot \dots \cdot \frac{n-2}{n} \cdot 1\right)^2} \\ &= \frac{1 \cdot 3 \cdot 5 \cdot \dots \cdot (n-3) \cdot (n+1) \cdot (n+2) \cdot \dots \cdot (2n-1)}{[2 \cdot 4 \cdot 6 \cdot \dots \cdot (n-2) \cdot n]^2} \\ &\geq \frac{(n+1) \cdot (n+3) \cdot \dots \cdot (2n-1)}{2 \cdot 4 \cdot \dots \cdot (n-4) \cdot (n-2)^2 \cdot n^2} \geq \frac{\sqrt{2}^n}{(n-2)n^2} \geq C \cdot 1.4^n\end{aligned}$$

for all  $n \in \{6, 8, 10, \dots\}$  and some constant  $C \in (0, \infty)$ .

Theory [1]: for uniformly spaced nodes  $\boxed{\lambda_{t_0, \dots, t_n}(I) \geq Ce^{n/2}}$  for all  $n \in \mathbb{N}$  and some constant  $C \in (0, \infty)$ . ◇

*Example 6.2.5 (Oscillating interpolation polynomial: Runge's counterexample).* Between the nodes the interpolation polynomial can oscillate excessively and overestimate the changes in the values: bad approximation of functions!

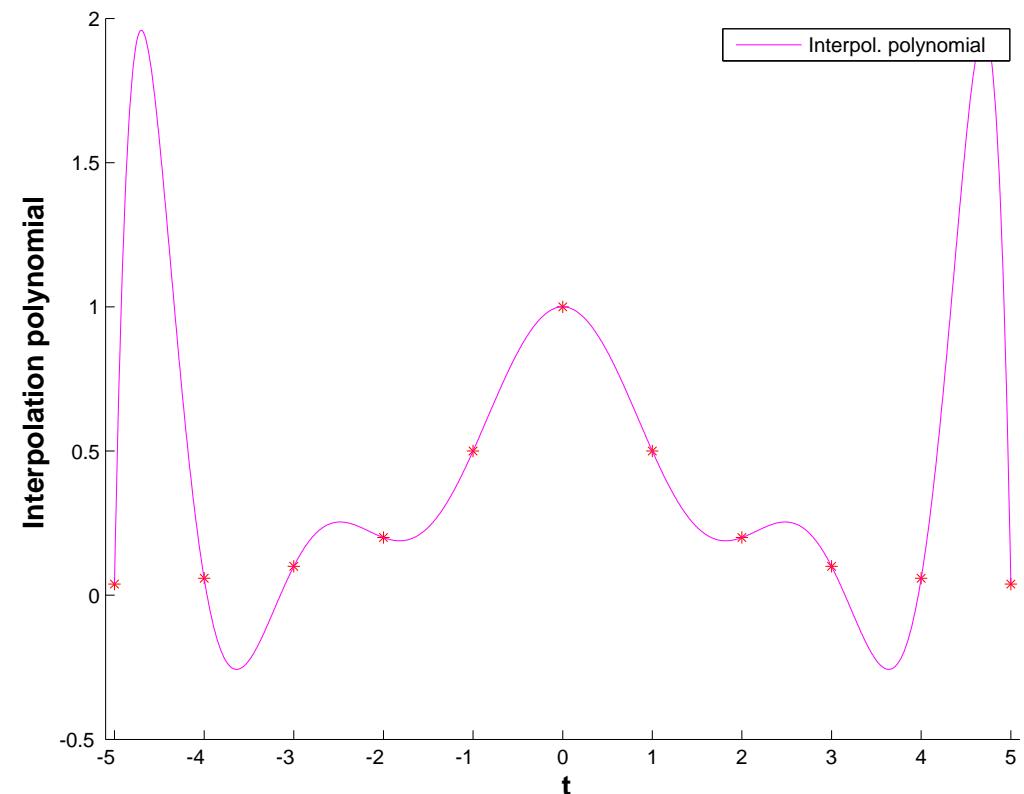
Let  $n \in \mathbb{N}$ . Interpolation polynomial with uniformly spaced nodes:

$$t_j := -5 + \frac{10j}{n}, \quad y_j = \frac{1}{1 + (t_j)^2}$$

for all  $j \in \{0, 1, \dots, n\}$ .

Plot  $n = 10 \rightarrow$

See Example 6.4.1.



Attention:  
strong oscillations of the interpolation polynomials of high degree on uniformly spaced nodes!



# 6.3 Polynomial Interpolation: Algorithms

## 6.3.1 Newton basis polynomials and divided differences

Drawback of the Lagrange basis: adding another data point affects *all* basis polynomials!

Alternative, “update friendly” method: **Newton basis** of  $\mathcal{P}_n$ .

**Definition 6.3.1.** Consider  $t_0, \dots, t_n \in \mathbb{R}$  with  $|\{t_0, \dots, t_n\}| = n + 1$ . Then define **Newton basis polynomials**  $N_0, N_1, \dots, N_n \in \mathcal{P}_n$  associated to  $(t_0, \dots, t_n)$  through

$$N_0(t) := 1 , \quad N_1(t) := (t - t_0) , \quad \dots , \quad N_n(t) := \prod_{i=0}^{n-1} (t - t_i) \quad (6.3.1)$$

for all  $t \in \mathbb{R}$ .

➤ LSE for polynomial interpolation problem in Newton basis:

Given  $(y_j)_{j \in \{0,1,\dots,n\}} \subset \mathbb{R}$ , find  $(a_j)_{j \in \{0,1,\dots,n\}} \subset \mathbb{R}$  such that

$$\forall j \in \{0, 1, \dots, n\}: \quad a_0 N_0(t_j) + a_1 N_1(t_j) + \dots + a_n N_n(t_j) = y_j .$$

$\Leftrightarrow$  triangular linear system

$$\begin{pmatrix} N_0(t_0) & N_1(t_0) & N_2(t_0) & \cdots & N_n(t_0) \\ N_0(t_1) & N_1(t_1) & N_2(t_1) & \cdots & N_n(t_1) \\ N_0(t_2) & N_1(t_2) & N_2(t_2) & \cdots & N_n(t_2) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ N_0(t_n) & N_1(t_n) & N_2(t_n) & \cdots & N_n(t_n) \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-1} \\ a_n \end{pmatrix} = \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_{n-1} \\ y_n \end{pmatrix},$$

i.e.,

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & (t_1 - t_0) & 0 & \cdots & \vdots \\ 1 & (t_2 - t_0) & (t_2 - t_0)(t_2 - t_1) & \cdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 1 & (t_n - t_0) & (t_n - t_0)(t_n - t_1) & \cdots & \prod_{i=0}^{n-1} (t_n - t_i) \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-1} \\ a_n \end{pmatrix} = \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_{n-1} \\ y_n \end{pmatrix}.$$

## Solution of the system with forward substitution:

$$a_0 = y_0 ,$$

$$a_1 = \frac{y_1 - a_0}{t_1 - t_0} = \frac{y_1 - y_0}{t_1 - t_0} ,$$

$$a_2 = \frac{y_2 - a_0 - (t_2 - t_0)a_1}{(t_2 - t_0)(t_2 - t_1)} = \frac{y_2 - y_0 - (t_2 - t_0)\frac{y_1 - y_0}{t_1 - t_0}}{(t_2 - t_0)(t_2 - t_1)} = \frac{\frac{y_2 - y_0}{t_2 - t_0} - \frac{y_1 - y_0}{t_1 - t_0}}{t_2 - t_1} ,$$

$$a_3 = \frac{y_3 - a_0 - (t_3 - t_0)a_1 - (t_3 - t_0)(t_3 - t_1)a_2}{(t_3 - t_0)(t_3 - t_1)(t_3 - t_2)} = \frac{\frac{y_3 - y_0}{t_3 - t_0} - \frac{y_1 - y_0}{t_1 - t_0} - (t_3 - t_1)a_2}{(t_3 - t_1)(t_3 - t_2)}$$

$$= \frac{\frac{y_3 - y_0}{t_3 - t_0} - \frac{y_1 - y_0}{t_1 - t_0} - \frac{y_2 - y_0}{t_2 - t_0} - \frac{y_1 - y_0}{t_1 - t_0}}{(t_3 - t_2)}$$

:

Simpler and more efficient algorithm using **divided differences**:

$$\begin{aligned} y[t_i] &= y_i \\ y[t_i, t_{i+1}, \dots, t_{i+k}] &= \frac{y[t_{i+1}, \dots, t_{i+k}] - y[t_i, \dots, t_{i+k-1}]}{t_{i+k} - t_i} \quad (\text{recursion}) \end{aligned} \quad (6.3.2)$$

Recursive calculation by **divided differences scheme**:

$$\begin{array}{c|ccccc} t_0 & y[t_0] & & & & \\ t_1 & & y[t_0, t_1] & & & \\ t_2 & & & y[t_0, t_1, t_2] & & \\ t_3 & & & & y[t_0, t_1, t_2, t_3], \\ \hline y[t_1] & & & & & \\ y[t_2] & & & & & \\ y[t_3] & & & & & \end{array}$$

Then

$$\forall t \in \mathbb{R}: \sum_{i=0}^n y_i \cdot L_i(t) = (l_{t_0, \dots, t_n}(y_0, \dots, y_n))(t) = \sum_{i=0}^n y[t_0, \dots, t_i] \cdot N_i(t)$$

and  $y[t_0, t_1, \dots, t_i] = y[t_{\pi(0)}, t_{\pi(1)}, \dots, t_{\pi(i)}]$

for all permutations  $\pi: \{0, 1, \dots, i\} \rightarrow \{0, 1, \dots, i\}$  and all  $i \in \{0, 1, \dots, n\}$  (see, e.g., Satz 2.2 and Korollar 2.1 in [44, Rannacher, Einführung in die Numerische Mathematik, Vorlesungsskriptum, 2006]).

## Code 6.2: Divided differences, recursive implementation

```

1 function y = divdiff(t,y)
2 n = length(y)-1;
3 if (n > 0)
4     y(1:n) = divdiff(t(1:n),y(1:n));
5     for j=0:n-1
6         y(n+1) = (y(n+1)-y(j+1)) / (t(n+1)-t(j+1));
7     end
8 end
```

## Code 6.3: Divided differences evaluation by modified Horner scheme

```

1 function p = evaldivdiff(t,y,x)
2 n = length(y)-1;
3 dd=divdiff(t,y);
4 p=dd(n+1);
5 for j=n:-1:1
6     p = (x-t(j)).*p+dd(j);
7 end
```

Computational effort:

- $O(n^2)$  for computation of divided differences,
- $O(n)$  for every single evaluation of  $p(t)$ .

## 6.3.2 Extrapolation to zero

Extrapolation is the same as interpolation but the evaluation point  $t \in \mathbb{R}$  is outside the interval  $[\min_{j \in \{0, \dots, n\}} t_j, \max_{j \in \{0, \dots, n\}} t_j]$ . W.l.o.g. assume  $t = 0$ .

Assumption: Let  $I \subset \mathbb{R}$  be an interval with  $0, t_0, \dots, t_n \in I$ , let  $A_1, \dots, A_n \in \mathbb{R}$  and let  $f: I \rightarrow \mathbb{R}$  be a continuous function with the **asymptotic expansion**

$$f(h) = f(0) + A_1 h^2 + A_2 h^4 + \dots + A_n h^{2n} + R(h)$$

where  $R: I \rightarrow \mathbb{R}$  is the **remainder function** satisfying  $R(h) = O(h^{(2n+2)})$  as  $h \rightarrow 0$  (i.e.,  $\limsup_{h \rightarrow 0} (|R(h)| h^{-(2n+2)}) < \infty$ ). This is, e.g., fulfilled when  $I = \mathbb{R}$  and when  $f$  is a smooth and even function.

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Problem: compute  $f(0) = \lim_{t \rightarrow 0} f(t)$  with prescribed precision, when the evaluation of the function  $y=f(t)$  is unstable for  $|t| \ll 1$ .

Solution via extrapolation to zero:

- ① evaluation of  $f(t_i)$  for  $i \in \{0, 1, \dots, n\}$ .

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- ②  $f(0) \approx p(0) = [I_{t_0, \dots, t_n}(f(t_0), \dots, f(t_n))](0)$  with  
 interpolation polynomial  $p = I_{t_0, \dots, t_n}(f(t_0), \dots, f(t_n)) \in \mathcal{P}_n$ .

*Example 6.3.4* (Numeric differentiation through extrapolation). Let  $x \in \mathbb{R}$  and let  $g: \mathbb{R} \rightarrow \mathbb{R}$  be a  $(2n+3)$ -times continuously differentiable function. Then define  $f: \mathbb{R} \rightarrow \mathbb{R}$  through

$$f(h) := \begin{cases} \frac{g(x+h) - g(x-h)}{2h} = g'(x) + \sum_{k=1}^n \frac{g^{(2k+1)}(x) h^{2k}}{(2k+1)!} + O(h^{2n+2}) & : h \neq 0 \\ g'(x) & : h = 0 \end{cases}.$$

Note that  $f(0) = \lim_{h \rightarrow 0} f(h) = g'(x)$ . We thus get  $g'(x) \approx [I_{t_0, \dots, t_n}(f(t_0), \dots, f(t_n))](0)$  for sufficiently small  $t_0, \dots, t_n \in (0, \infty)$ . ◇

## 6.4 Interpolation Error Estimates

Consider bounded and closed interval  $I \subset \mathbb{R}$  and Lagrangian polynomial interpolation with nodes  $t_0, \dots, t_n \in I$  satisfying  $t_0 < t_1 < \dots < t_n$ .

Goal: estimate of the **interpolation error** norm  $\|f - I_{t_0, \dots, t_n}(f(t_0), \dots, f(t_n))\|$  where  $\|\cdot\|$  is a norm on  $C(I, \mathbb{R})$  and where  $f \in C(I, \mathbb{R})$ .

**Definition 6.4.1** (Error polynomial). Let  $s_0, \dots, s_n \in \mathbb{R}$ . Then we denote by  $e_{s_0, \dots, s_n} \in \mathcal{P}_{n+1}$  defined through

$$e_{s_0, \dots, s_n}(t) := (t - s_0) \cdot (t - s_1) \cdot \dots \cdot (t - s_n) \quad (6.4.1)$$

for all  $t \in \mathbb{R}$  the error polynomial associated to  $(s_0, \dots, s_n)$

**Theorem 6.4.2** (Representation of interpolation error).

Let  $f \in C^{n+1}(I, \mathbb{R})$  and  $t \in I$ . Then there exists  $\tau_t \in (\min\{t, t_0, \dots, t_n\}, \max\{t, t_0, \dots, t_n\})$  such that

$$f(t) - \left( I_{t_0, \dots, t_n}(f(t_0), \dots, f(t_n)) \right)(t) = \frac{f^{(n+1)}(\tau_t)}{(n+1)!} \cdot \prod_{j=0}^n (t - t_j) = \frac{f^{(n+1)}(\tau_t) \cdot e_{t_0, \dots, t_n}(t)}{(n+1)!}.$$

**Corollary 6.4.3** (Estimate of interpolation error). Let  $f \in C^{n+1}(I, \mathbb{R})$ . Then

$$\|f - I_{t_0, \dots, t_n}(f(t_0), \dots, f(t_n))\|_{\infty, I} \leq \frac{\|f^{(n+1)}\|_{\infty, I} \|e_{t_0, \dots, t_n}\|_{\infty, I}}{(n+1)!}.$$

Interpolation error estimate requires smoothness!

*Example 6.4.1* (Runge's example). Polynomial interpolation for  $I = [-5, 5]$  and  $f: I \rightarrow \mathbb{R}$  given by  $f(t) = \frac{1}{1+t^2}$  for all  $t \in I$  with  $n = 10$  and equispaced nodes

$$t_j := -5 + \frac{10j}{n}, \quad y_j := \frac{1}{1 + (t_j)^2}, \quad j \in \{0, 1, \dots, n\}.$$

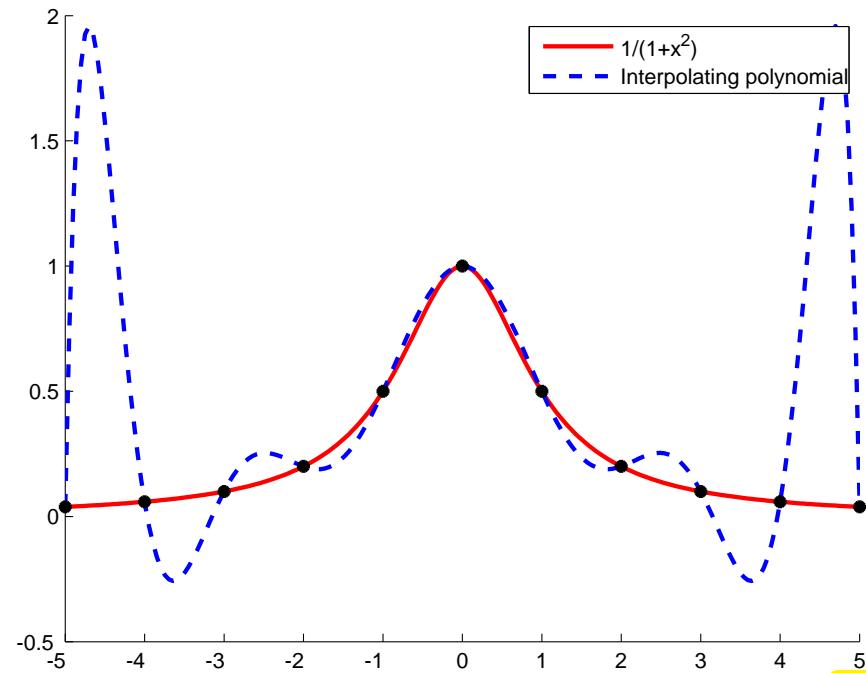


Fig. 37

Interpolating polynomial,  $n = 10$



# 6.5 Chebychev Interpolation

## 6.5.1 Motivation and definition

W.l.o.g. let  $I = [-1, 1] \subset \mathbb{R}$ . Then Corollary 6.4.3 ensures that

$$\|f - I_{t_0, \dots, t_n}(f(t_0), \dots, f(t_n))\|_{\infty, [-1, 1]} \leq \frac{\|f^{(n+1)}\|_{\infty, [-1, 1]} \|e_{t_0, \dots, t_n}\|_{\infty, [-1, 1]}}{(n+1)!} \quad (6.5.1)$$

for all  $-1 \leq t_0 < \dots < t_n \leq 1$  and all  $f \in C^{n+1}([-1, 1], \mathbb{R})$ .

Goal: Find  $-1 \leq t_0 < \dots < t_n \leq 1$  such that  $\|e_{t_0, \dots, t_n}\|_{\infty, [-1, 1]}$  is minimal!

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Recall the inverse function  $\arccos: [-1, 1] \rightarrow [0, \pi]$  of  
the function  $[0, \pi] \ni x \mapsto \cos(x) \in [-1, 1]$ . Note that

$$\forall t \in [-1, 1], x \in [0, \pi]: \quad \cos(\arccos(t)) = t \quad \text{and} \quad \arccos(\cos(x)) = x. \quad (6.5.2)$$

**Definition 6.5.1** (Chebychev polynomial).

By  $T_n: \mathbb{R} \rightarrow \mathbb{R} \in \mathcal{P}_n$  we denote the  $n$ -th **Chebychev polynomial** which is characterised through  $T_n(t) = \cos(n \arccos(t))$  for all  $t \in [-1, 1]$ .

The well-definedness of Definition 6.5.1 is given in the proof of the next lemma.

**Lemma 6.5.2** (Chebychev polynomial). *It holds that*

$$T_n(t) = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} t^{(n-2k)} (t^2 - 1)^k \quad (6.5.3)$$

for all  $t \in \mathbb{R}$ .

$$\begin{aligned}
\cos(n \arccos(t)) &= \operatorname{Re}(e^{i \cdot n \cdot \arccos(t)}) = \operatorname{Re}\left([e^{i \arccos(t)}]^n\right) \\
&= \operatorname{Re}\left(\left[\cos(\arccos(t)) + i \sin(\arccos(t))\right]^n\right) = \operatorname{Re}\left([t + i\sqrt{1-t^2}]^n\right) \\
&= \operatorname{Re}\left(\sum_{k=0}^n \binom{n}{k} t^{(n-k)} (i\sqrt{1-t^2})^k\right) = \sum_{k=0}^n \binom{n}{k} t^{(n-k)} \operatorname{Re}(i^k) (1-t^2)^{\frac{k}{2}} \quad (6.5.4) \\
&= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} t^{(n-2k)} \operatorname{Re}(i^{2k}) (1-t^2)^k = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} t^{(n-2k)} (-1)^k (1-t^2)^k
\end{aligned}$$

for all  $t \in [-1, 1]$ . This completes the proof of Lemma 6.5.2. □

Note that Lemma 6.5.2 implies that for every  $t \in \mathbb{R}$  that

$$T_0(t) = 1, \quad T_1(t) = t, \quad T_2(t) = \binom{2}{0} t^2 + \binom{2}{2} (t^2 - 1) = t^2 + t^2 - 1 = 2t^2 - 1, \quad \dots.$$

Moreover, observe that Definition 6.5.1 ensures that

$$T_n(1) = \cos(0) = 1 \quad \text{and} \quad |T_n(t)| \leq 1 \quad (6.5.5)$$

for all  $t \in [-1, 1]$  and hence that  $\|T_n\|_{\infty, [-1, 1]} = 1$ .

**Lemma 6.5.3** (3-term recursion for Chebychev polynomials). *It holds that*

$$T_{k+1}(t) = 2tT_k(t) - T_{k-1}(t), \quad T_0(t) = 1, \quad T_1(t) = t \quad (6.5.6)$$

for all  $t \in \mathbb{R}$  and all  $k \in \mathbb{N}$ .

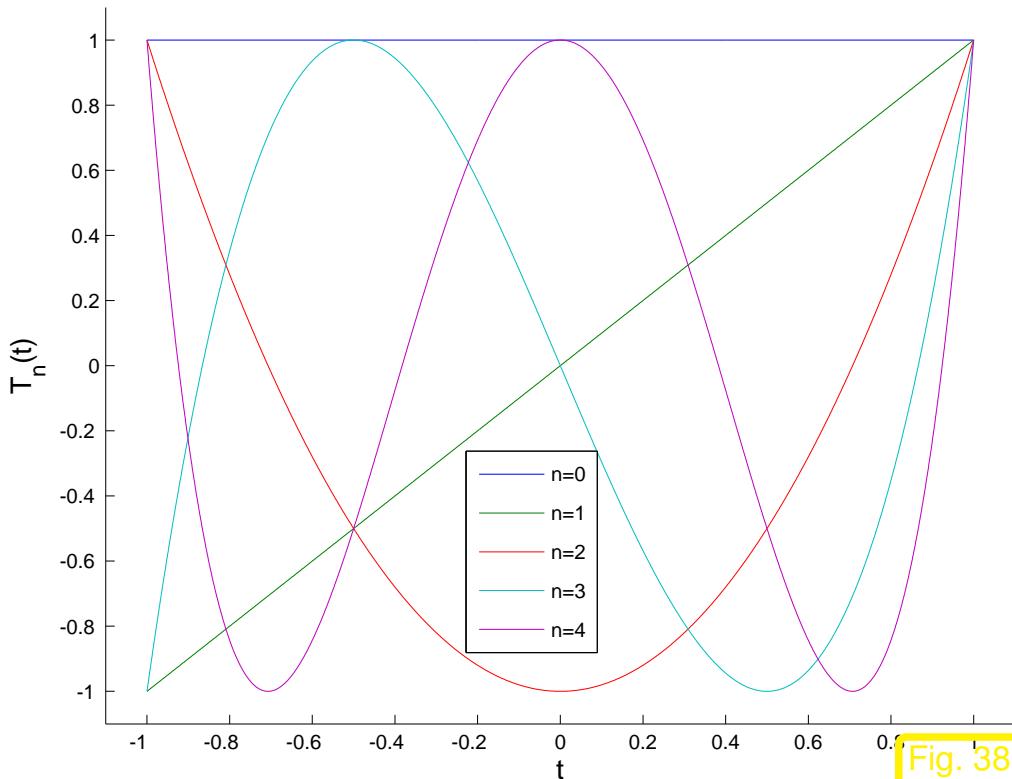


Fig. 38

Chebychev polynomials  $T_0, \dots, T_4$

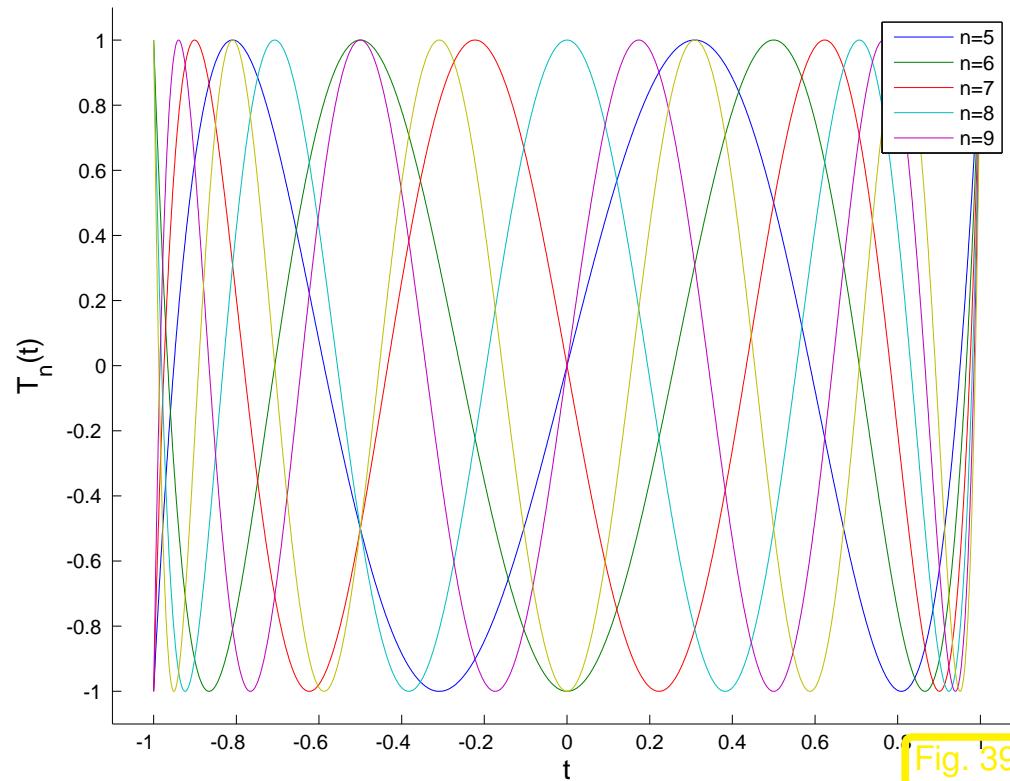


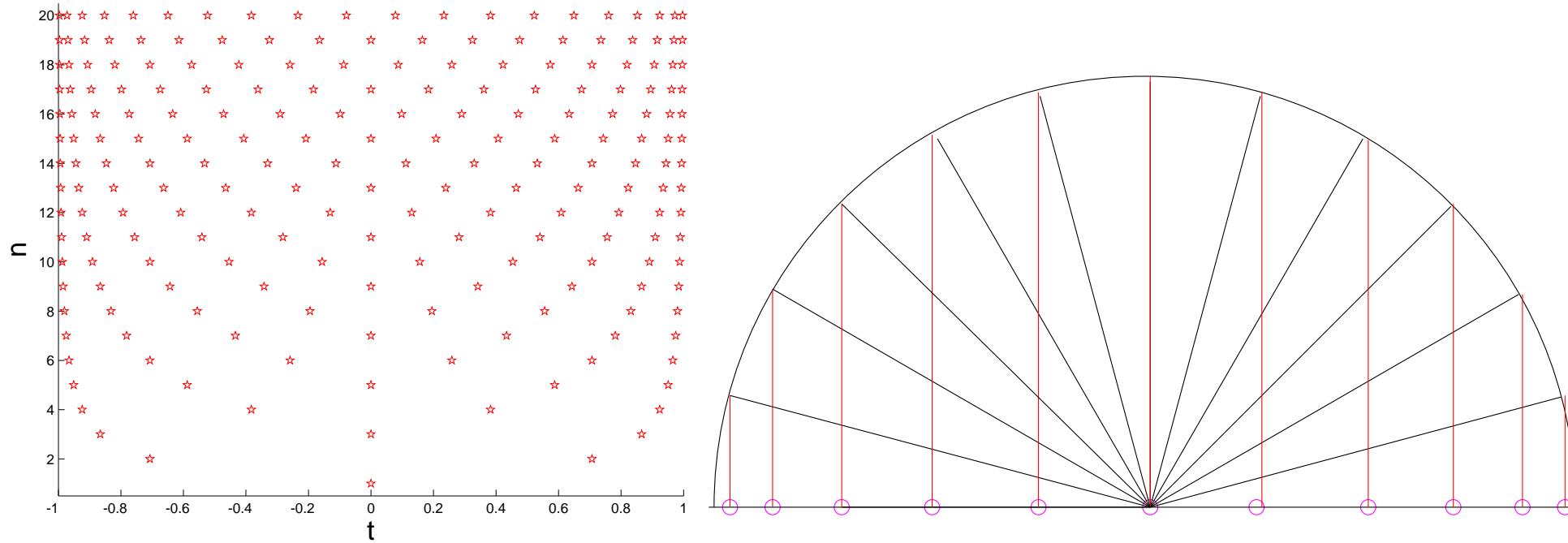
Fig. 39

Chebychev polynomials  $T_5, \dots, T_9$

**Definition 6.5.4** (Zeros of Chebychev polynomials). Let  $n \in \mathbb{N}$ . Then we denote by  $\tau_1^{(n)} > \tau_2^{(n)} > \dots > \tau_n^{(n)}$  the zeros of the  $n$ -th Chebychev polynomial  $T_n$ , i.e., we define

$$\forall k \in \{1, 2, \dots, n\}: \quad \tau_k^{(n)} := \cos\left(\left[\frac{k - \frac{1}{2}}{n}\right]\pi\right) \in (-1, 1) \quad (6.5.7)$$

Chebychev nodes  $\tau_k^{(n)}$ ,  $k \in \{1, \dots, n\}$ , from (6.5.7):



Next we note that it holds that

$$\forall t \in \mathbb{R}, k \in \mathbb{N}: \quad \frac{T_k(t)}{2^{(k-1)}} = (t - \tau_1^{(k)}) \cdot (t - \tau_2^{(k)}) \cdot \dots \cdot (t - \tau_k^{(k)}) . \quad (6.5.9)$$

**Theorem 6.5.5** (Minimality property of the zeros of the Chebychev polynomials). *It holds that*

$$\left\| \frac{T_{n+1}}{2^n} \right\|_{\infty, [-1,1]} = \left\| e_{\tau_1^{(n+1)}, \dots, \tau_{n+1}^{(n+1)}} \right\|_{\infty, [-1,1]} = \min \left\{ \begin{array}{c} \|e_{t_0, \dots, t_n}\|_{\infty, [-1,1]} : \\ -1 \leq t_0 < \dots < t_n \leq 1 \end{array} \right\} = \frac{1}{2^n} .$$

See [3, Section 7.1.4] for the proof of Theorem 6.5.5.

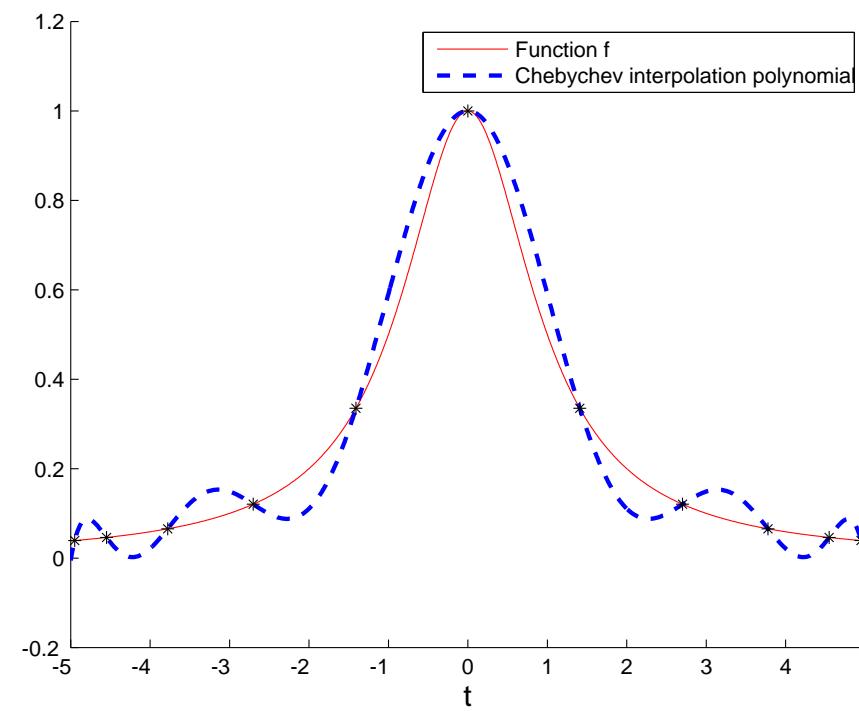
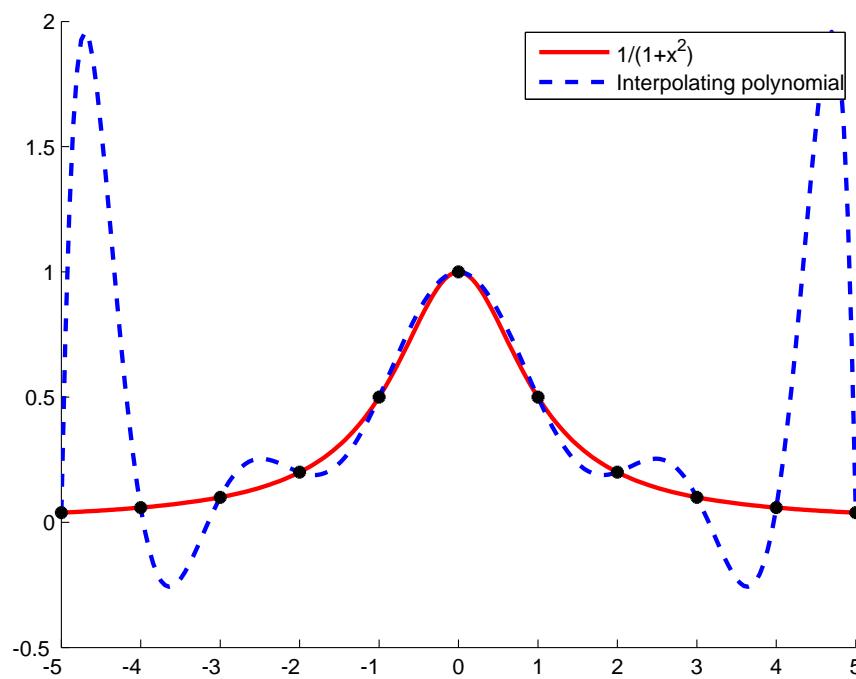
## 6.5.2 Chebychev interpolation error estimates

Combining Corollary 6.4.3 and Theorem 6.5.5 shows for every  $f \in C^{n+1}([-1, 1], \mathbb{R})$  that

$$\left\| f - I_{\tau_{n+1}^{(n+1)}, \dots, \tau_1^{(n+1)}}(f(\tau_{n+1}^{(n+1)}), \dots, f(\tau_1^{(n+1)})) \right\|_{\infty, [-1,1]} \leq \frac{\|f^{(n+1)}\|_{\infty, [-1,1]}}{(n+1)! 2^n} . \quad (6.5.13)$$

### Example 6.5.3 (Polynomial interpolation: Chebychev nodes versus equidistant nodes).

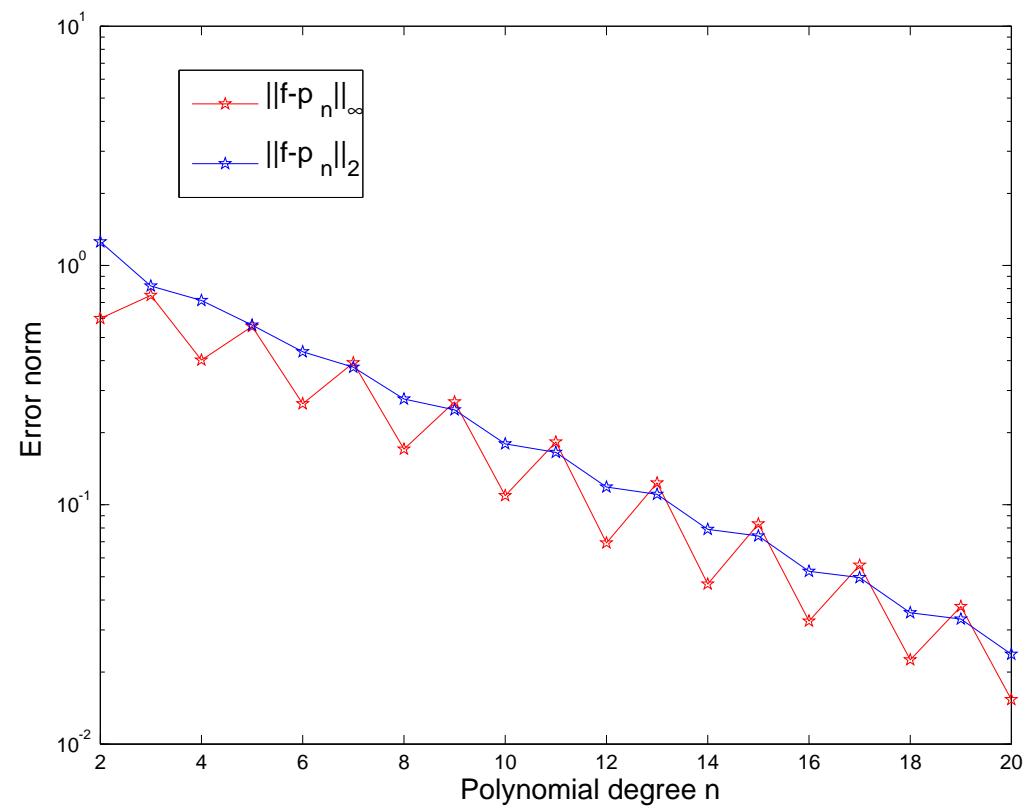
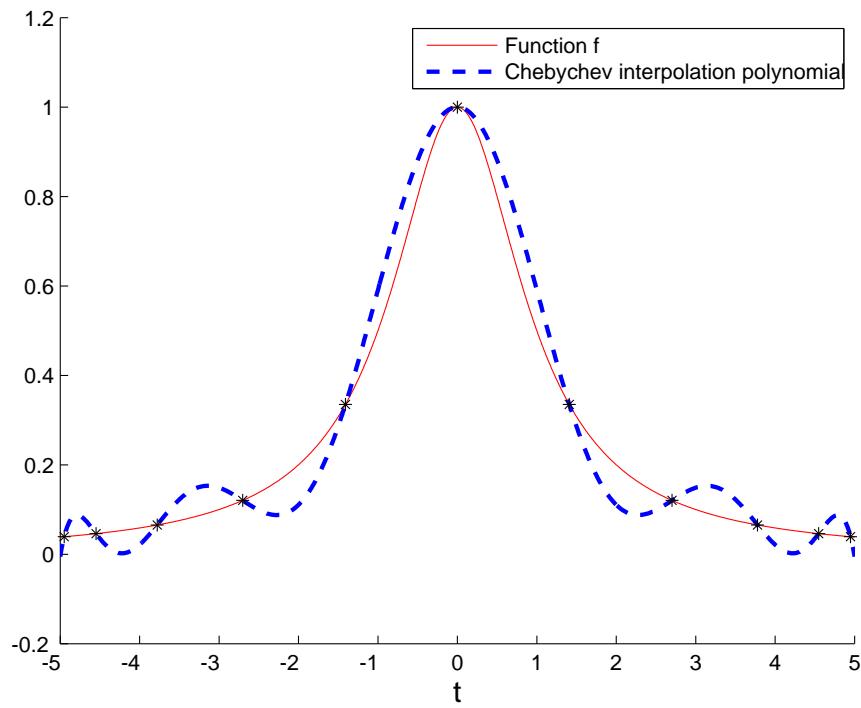
Runge's function  $f(t) = \frac{1}{1+t^2}$ , see Ex. 6.4.1, polynomial interpolation based on uniformly spaced nodes and Chebychev nodes:



## Example 6.5.5 (Chebychev interpolation error).

$$f(t) = (1 + t^2)^{-1}, \quad t \in I = [-5, 5] \text{ (see Ex. 6.4.1)}$$

Interpolation with  $n = 10$  Chebychev nodes (plot on the left).



### 6.5.3 Chebychev interpolation: computational aspects

Note that  $T_0, T_1, \dots, T_n$  form a basis in the  $(n + 1)$ -dimensional  $\mathbb{R}$ -vector  $\mathcal{P}_n$ .

Hence, for every  $p \in \mathcal{P}_n$  there exist  $\alpha_0, \alpha_1, \dots, \alpha_n \in \mathbb{R}$  such that

$$p = \sum_{j=0}^n \alpha_j T_j. \quad (6.5.15)$$

Equation (6.5.15) is also called *Chebychev expansion* (of  $p$ ).

*Remark 6.5.6* (Fast evaluation of Chebychev expansion).  $\rightarrow [10, \text{Alg. 32.1}]$  If  $n \in \{2, 3, \dots\}$  and  $x \in \mathbb{R}$ , then we use the 3-term recursion (see (6.5.6))

$$T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x)$$

to rewrite (6.5.15) as

$$p(x) = \sum_{j=0}^n \alpha_j T_j(x) = \sum_{j=0}^{n-1} \tilde{\alpha}_j T_j(x) \quad \text{where} \quad \tilde{\alpha}_j = \begin{cases} \alpha_{n-1} + 2x\alpha_n & : j = n-1 \\ \alpha_{n-2} - \alpha_n & : j = n-2 \\ \alpha_j & : 0 \leq j \leq n-3 \end{cases} \quad (6.5.16)$$



recursive algorithm, see Code 6.4.

Code 6.4: Recursive evaluation of Chebychev expansion (6.5.15)

```
function y = recclenshaw(a, x)
```

```

2 % Recursive evaluation of a polynomial  $p = \sum_{j=1}^{n+1} a_j T_{j-1}$  at point  $x$ , see (6.5.16)
3 % The coefficients  $a_j$  have to be passed in a row vector.
4 n = length(a)-1;
5 if (n<2), y = a(1)+x*a(2);
6 else
7   y = recclenshaw([a(1:n-2), a(n-1)-a(n+1), a(n)+2*x*a(n+1)], x);
8 end

```

► Computational effort:  $O(n)$  for evaluation at point  $x \in \mathbb{R}$ . △

Given:  $y_0, y_1, \dots, y_n \in \mathbb{R}$ . Aim: Find  $\alpha_0, \alpha_1, \dots, \alpha_n \in \mathbb{R}$  such that

$$\forall k \in \{0, 1, \dots, n\}: \quad p(\tau_{k+1}^{(n+1)}) = \sum_{j=0}^n \alpha_j T_j(\tau_{k+1}^{(n+1)}) = y_k. \quad (6.5.17)$$

Trick: Define 1-periodic function  $q: \mathbb{R} \rightarrow \mathbb{R}$  through

$$q(s) := p(\cos(2\pi s)) = \sum_{j=0}^n \alpha_j T_j(\cos(2\pi s)) \quad (6.5.18)$$

for all  $s \in \mathbb{R}$ . Then (6.5.17) reads as

$$\forall k \in \{0, 1, \dots, n\}: \quad q\left(\frac{2k+1}{4(n+1)}\right) = q\left(\frac{k+\frac{1}{2}}{2(n+1)}\right) = p(\tau_{k+1}^{(n+1)}) = y_k. \quad (6.5.19)$$

The symmetry property  $q(s) = q(1 - s)$  for all  $s \in \mathbb{R}$  hence shows

$$\forall k \in \{0, 1, \dots, n\}: \quad q\left(1 - \frac{2k+1}{4(n+1)}\right) = y_k. \quad (6.5.20)$$

Next define  $\mathbf{z} = (z_0, z_1, \dots, z_{2n+1}) \in \mathbb{R}^{2(n+1)}$  through

$$\forall k \in \{0, 1, \dots, 2n+1\}: \quad z_k := \begin{cases} \exp\left(\frac{-\pi i n k}{(n+1)}\right) y_k & : k \leq n \\ \exp\left(\frac{-\pi i n k}{(n+1)}\right) y_{2n+1-k} & : k \geq n+1 \end{cases}. \quad (6.5.21)$$

Combining (6.5.19) and (6.5.20) then shows

$$\forall k \in \{0, 1, \dots, 2n+1\}: \quad q\left(\frac{k}{2(n+1)} + \frac{1}{4(n+1)}\right) = \exp\left(\frac{\pi i n k}{(n+1)}\right) z_k. \quad (6.5.22)$$

In addition, observe that the definition of  $q$  ensures that

$$\forall s \in [0, \frac{1}{2}]: \quad q(s) = p(\cos(2\pi s)) = \sum_{j=0}^n \alpha_j T_j(\cos(2\pi s)) = \sum_{j=0}^n \alpha_j \cos(2\pi j s) \quad (6.5.23)$$

and this implies that

$$\begin{aligned} \forall s \in \mathbb{R}: \quad q(s) &= p(\cos(2\pi s)) = \sum_{j=0}^n \alpha_j T_j(\cos(2\pi s)) = \sum_{j=0}^n \alpha_j \cos(2\pi j s) \\ &= \sum_{j=0}^n \frac{\alpha_j}{2} \left( \exp(2\pi i j s) + \exp(-2\pi i j s) \right) =: \sum_{j=-n}^{n+1} \beta_j \exp(-2\pi i j s) \end{aligned} \quad (6.5.24)$$

where  $\beta_{-n}, \dots, \beta_{n+1} \in \mathbb{R}$  are defined through

$$\forall j \in \{-n, -n+1, \dots, n+1\}: \quad \beta_j := \begin{cases} 0 & : j = n+1 \\ \frac{\alpha_j}{2} & : j \in \{1, \dots, n\} \\ \alpha_0 & : j = 0 \\ \frac{\alpha_{-j}}{2} & : j \in \{-n, \dots, -1\} \end{cases}. \quad (6.5.25)$$

This and (6.5.22) then show that

$$\forall 0 \leq k \leq 2n+1: q\left(\frac{k}{2(n+1)} + \frac{1}{4(n+1)}\right) = \sum_{j=-n}^{n+1} \left( \beta_j \exp\left(-\frac{2\pi i j}{4(n+1)}\right) \right) \exp\left(-\frac{2\pi i j k}{2(n+1)}\right) = \exp\left(\frac{\pi i n k}{(n+1)}\right) z_k$$

and hence

$$\forall 0 \leq k \leq 2n+1: \quad \sum_{j=0}^{2n+1} \left( \beta_{(j-n)} \exp\left(-\frac{2\pi i (j-n)}{4(n+1)}\right) \right) \underbrace{\exp\left(-\frac{2\pi i j k}{2(n+1)}\right)}_{\omega_{2(n+1)}^{kj}} = z_k$$

and therefore

$$\mathbf{F}_{2(n+1)} \mathbf{c} = \mathbf{z} \quad \text{where}$$

$$\mathbf{c} := \left( \beta_{-n} \exp\left(\frac{2\pi i n}{4(n+1)}\right), \beta_{1-n} \exp\left(\frac{2\pi i (n-1)}{4(n+1)}\right), \dots, \beta_{n+1} \exp\left(\frac{2\pi i (-n-1)}{4(n+1)}\right) \right) \in \mathbb{C}^{2(n+1)}. \quad (6.5.26)$$



solve (6.5.26) with inverse discrete Fourier transform, see 5.2:

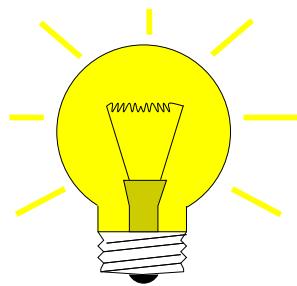
Code 6.5: Efficient computation of Chebychev expansion coefficient of Chebychev interpolant

```

1 function a = chebexp(y)
2 % Efficiently compute coefficients  $\alpha_j$  in the Chebychev expansion
3 %  $p = \sum_{j=0}^n \alpha_j T_j$  of  $p \in \mathcal{P}_n$  based on values  $y_k$ ,
4 %  $k = 0, \dots, n$ , in Chebychev nodes  $t_k$ ,  $k = 0, \dots, n$ . These values are
5 % passed in the row vector y.
6 n = length(y) - 1; % degree of polynomial
7 % create vector z by wrapping and componentwise scaling
8 z = exp(-pi*i*n/(n+1)*(0:2*n+1)).*[y, y(end: -1:1)]; % r.h.s. vector
9 c = ifft(z); % Solve linear system (6.5.26) with effort  $O(n \log n)$ 
10 b = real(exp(0.5 * pi*i/(n+1)*(-n:n+1)).*c); % recover  $\beta_j$ , see (6.5.26)
11 a = [b(n+1), 2*b(n+2:2*n+1)]; % recover  $\alpha_j$ , see (6.5.24)

```

## 6.6 Trigonometric Interpolation



Idea (J. Fourier 1822): Approximation of a function not by usual polynomials but by trigonometrical polynomials = partial sum of a Fourier series

## 6.6.1 Trigonometric Polynomials

**Definition 6.6.1** (Trigonometric polynomials). A function  $T: \mathbb{R} \rightarrow \mathbb{C}$  is called *trigonometric polynomial* if there exist a  $k \in \mathbb{N}_0$  and complex numbers  $\gamma_{-k}, \gamma_{1-k}, \dots, \gamma_k \in \mathbb{C}$  such that

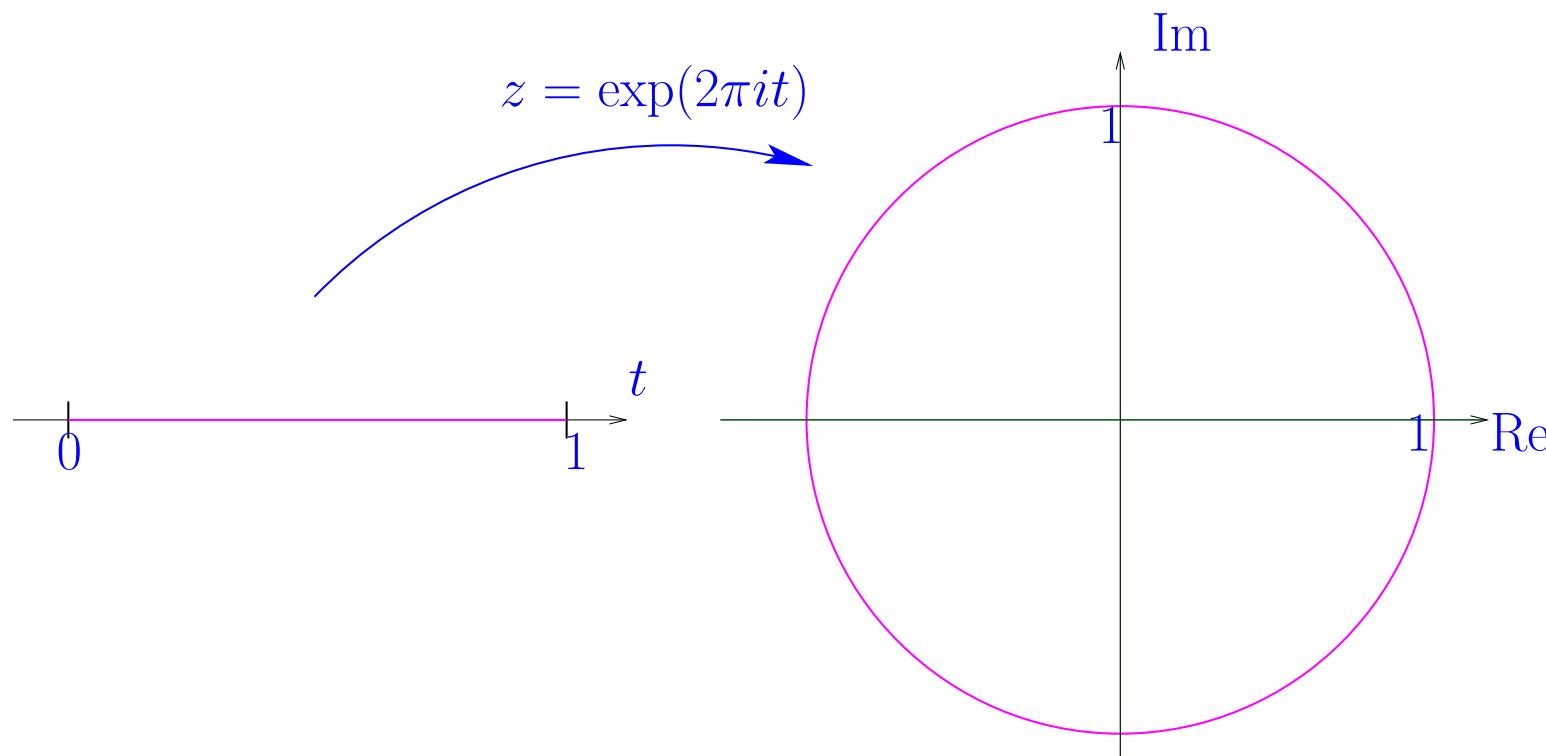
$$\forall t \in \mathbb{R}: \quad T(t) = \sum_{j=-k}^k \gamma_j e^{2\pi i j t}. \quad (6.6.1)$$

In that case the number  $\deg(T) := \max(\{l \in \{0, 1, \dots, k\}: |\gamma_l| + |\gamma_{-l}| \neq 0\} \cup \{-\infty\})$  is called *degree* of  $T$  and if additionally  $\deg(T) \geq 0$ , then the complex numbers  $\gamma_{-\deg(T)}, \gamma_{1-\deg(T)}, \dots, \gamma_{\deg(T)}$  are called *coefficients* of  $T$ . For every  $k \in \mathbb{N}_0$  we denote by  $\mathcal{T}_k$  the set of all trigonometric polynomials of degree  $\leq k$ , that is,

$$\mathcal{T}_k := \left\{ \mathbb{R} \ni t \mapsto \sum_{j=-k}^k \gamma_j e^{2\pi i j t} \in \mathbb{C}: \gamma_{-k}, \gamma_{1-k}, \dots, \gamma_k \in \mathbb{C} \right\}.$$

Moreover, for every  $j \in \mathbb{Z}$  we denote by  $w_j: \mathbb{R} \rightarrow \mathbb{C}$  the trigonometric polynomial given by  $\forall t \in \mathbb{R}: w_j(t) = e^{2\pi i j t}$ .

Parametrisation:  $[0, 1[ \xrightarrow[t \mapsto z]{z = e^{2\pi it}} \mathbb{S}^1 := \{z \in \mathbb{C}: |z| = 1\}$ .



For  $p \in [1, \infty)$  we denote in the following by  $L^p((0, 1), \mathbb{C})$  the  $\mathbb{C}$ -vector space of equivalence classes of  $p$ -integrable functions from  $(0, 1)$  to  $\mathbb{C}$ . For simplicity we do in the following not distinguish between a  $p$ -integrable function from  $(0, 1)$  to  $\mathbb{C}$  and its equivalence class in  $L^p((0, 1), \mathbb{C})$ . Moreover, in the following we denote by  $\|\cdot\|_{L^2((0,1),\mathbb{C})}$  and by  $\langle \cdot, \cdot \rangle_{L^2((0,1),\mathbb{C})}$

the norm and the scalar product in  $L^2((0, 1), \mathbb{C})$ , i.e., for every  $f, g \in L^2((0, 1), \mathbb{C})$  we define

$$\|f\|_{L^2((0,1),\mathbb{C})} := \left[ \int_0^1 |f(t)|^2 dt \right]^{1/2} \quad \text{and} \quad \langle f, g \rangle_{L^2((0,1),\mathbb{C})} := \int_0^1 \overline{f(t)} g(t) dt .$$

**Lemma 6.6.2** (Basis functions of trigonometric polynomials). *For every  $k \in \mathbb{N}_0$  the functions  $w_{-k}, w_{1-k}, \dots, w_k \in \mathcal{T}_k$  form a basis of the  $\mathbb{C}$ -vector space  $\mathcal{T}_k$ . In addition, the functions  $w_j$ ,  $j \in \mathbb{Z}$ , are orthonormal in  $L^2((0, 1), \mathbb{C})$ , i.e., it holds that*

$$\forall j, k \in \mathbb{Z}: \quad \langle w_j, w_k \rangle_{L^2((0,1),\mathbb{C})} = \int_0^1 \overline{w_j(t)} w_k(t) dt = \begin{cases} 1 & : j = k \\ 0 & : j \neq k \end{cases} .$$

**Remark 6.6.1** (Periodicity of trigonometric polynomials). A trigonometric polynomial  $T: \mathbb{R} \rightarrow \mathbb{C}$  is a 1-periodic function, i.e., it holds for every  $t \in \mathbb{R}$  and every  $l \in \mathbb{Z}$  that  $T(t) = T(t + l)$ .

The next lemma provides a suitable characterization of trigonometric polynomials; cf., e.g.,

[http://en.wikipedia.org/wiki/Trigonometric\\_polynomial](http://en.wikipedia.org/wiki/Trigonometric_polynomial).

**Lemma 6.6.3** (Representation of trigonometric polynomials). A function  $T: \mathbb{R} \rightarrow \mathbb{C}$  is a trigonometric polynomial if and only if there exist a  $k \in \mathbb{N}_0$  and complex numbers  $a_0, a_1, \dots, a_k, b_1, \dots, b_k \in \mathbb{C}$  such that

$$\forall t \in \mathbb{R}: \quad T(t) = a_0 + \left[ \sum_{j=1}^k a_j \cos(2\pi jt) \right] + \left[ \sum_{j=1}^k b_j \sin(2\pi jt) \right]. \quad (6.6.2)$$

Moreover, if  $T: \mathbb{R} \rightarrow \mathbb{C}$  is a trigonometric polynomial of degree  $n$  and with coefficients  $\gamma_{-n}, \gamma_{1-n}, \dots, \gamma_n \in \mathbb{C}$ , then

$$\forall t \in \mathbb{R}: \quad T(t) = \gamma_0 + \left[ \sum_{j=1}^k (\gamma_j + \gamma_{-j}) \cos(2\pi jt) \right] + \left[ \sum_{j=1}^k i(\gamma_j - \gamma_{-j}) \sin(2\pi jt) \right]. \quad (6.6.3)$$

**Corollary 6.6.4** (Real valued trigonometric polynomials). Let  $T: \mathbb{R} \rightarrow \mathbb{C}$  be a trigonometric polynomial of degree  $n$  and with coefficients  $\gamma_{-n}, \gamma_{1-n}, \dots, \gamma_n \in \mathbb{C}$ . Then  $T$  is real valued if and only if it holds for every  $j \in \{0, 1, \dots, n\}$  that  $\overline{\gamma_j} = \gamma_{-j}$ .

## 6.6.2 Fourier coefficients and Fourier series

**Definition 6.6.5 (Fourier coefficients).** If  $f \in L^1((0, 1), \mathbb{C})$ , then we define the *Fourier coefficient function*  $\widehat{f}: \mathbb{Z} \rightarrow \mathbb{C}$  through

$$\widehat{f}(k) := \int_0^1 \overline{w_k(t)} f(t) dt = \int_0^1 w_{-k}(t) f(t) dt = \int_0^1 e^{-2\pi i k t} f(t) dt \quad (6.6.4)$$

for all  $k \in \mathbb{Z}$ . In addition, if  $f: \mathbb{R} \rightarrow \mathbb{C}$  is a 1-periodic function with  $f|_{(0,1)} \in L^1((0, 1), \mathbb{C})$ , then we define the *Fourier coefficient function*  $\widehat{f}: \mathbb{Z} \rightarrow \mathbb{C}$  through  $\widehat{f} := \widehat{f|_{(0,1)}}$ .

**Example 6.6.2.** Let  $j \in \mathbb{Z}$  and let  $f: \mathbb{R} \rightarrow \mathbb{C}$  be given by  $f := w_j$ . Then it holds that

$$\widehat{f}(k) = \int_0^1 f(t) w_{-k}(t) dt = \int_0^1 w_j(t) w_{-k}(t) dt = \begin{cases} 1 & : j = k \\ 0 & : j \neq k \end{cases} \quad (6.6.5)$$

for all  $k \in \mathbb{Z}$ .

*Example 6.6.3.* Let  $a, b \in (0, 1)$  with  $a < b$  and consider the function  $f: (0, 1) \rightarrow \mathbb{C}$  given by

$$f(t) = \begin{cases} 1 & : t \in [a, b] \\ 0 & : \text{else} \end{cases} \quad (6.6.6)$$

for all  $t \in (0, 1)$ . Then

$$\hat{f}(k) = \begin{cases} b - a & : k = 0 \\ \frac{1}{k\pi} e^{-i\pi k(a+b)} \sin(k\pi(b-a)) & : k \neq 0 \end{cases} \quad (6.6.7)$$

for all  $k \in \mathbb{Z}$ .

Next we formulate a convergence theorem for Fourier series (might be known from lectures in Analysis, Mathematical Methods of Physics, etc.).

**Theorem 6.6.6** ( $L^2$ -convergence of the Fourier series). *Let  $f \in L^2((0, 1), \mathbb{C})$ . Then*

$$f = \sum_{k=-\infty}^{\infty} \hat{f}(k) w_k = \sum_{k \in \mathbb{Z}} \hat{f}(k) w_k = \sum_{k \in \mathbb{Z}} \hat{f}(k) e^{2\pi i k (\cdot)} \quad \text{in } L^2((0, 1), \mathbb{C}), \quad \text{i.e.,}$$

$$\lim_{\substack{A \nearrow \mathbb{Z} \\ |A| < \infty}} \left\| f - \sum_{k \in A} \hat{f}(k) w_k \right\|_{L^2((0, 1), \mathbb{C})} = 0$$

**Remark 6.6.4.** In view of this theorem, we may think of a function  $f \in L^2((0, 1), \mathbb{C})$  in two ways: once in the *time (or space) domain*  $(0, 1) \ni t \mapsto f(t) \in \mathbb{C}$  and once in the *frequency domain*  $\mathbb{Z} \ni k \mapsto \widehat{f}(k) \in \mathbb{C}$ .

**Lemma 6.6.7 (Parseval's identity (Isometry property)).** Let  $f \in L^2((0, 1), \mathbb{C})$ . Then

$$\sum_{k=-\infty}^{\infty} |\widehat{f}(k)|^2 = \|f\|_{L^2((0,1),\mathbb{C})}^2. \quad (6.6.8)$$

**Lemma 6.6.8 (Derivative and Fourier coefficients).** Let  $f: \mathbb{R} \rightarrow \mathbb{C}$  be a continuously differentiable and 1-periodic function. Then  $\widehat{f}'(k) = 2\pi i k \widehat{f}(k)$  for all  $k \in \mathbb{Z}$ .

**Remark 6.6.5.** Let  $f: \mathbb{R} \rightarrow \mathbb{C}$  be an  $n$ -times continuously differentiable and 1-periodic function. Lemma 6.6.7 and Lemma 6.6.8 then show that

$$\infty > \|f^{(n)}\|_{L^2((0,1),\mathbb{C})}^2 = \sum_{k=-\infty}^{\infty} |\widehat{f^{(n)}}(k)|^2 = (2\pi)^{2n} \left[ \sum_{k=-\infty}^{\infty} k^{2n} |\widehat{f}(k)|^2 \right]. \quad (6.6.9)$$

and this implies that  $\sup_{k \in \mathbb{Z}} (|k|^n |\widehat{f}(k)|) < \infty$ .

The smoothness of a function directly reflects in the quick decay of its Fourier coefficients.

**Lemma 6.6.9.** There exist real numbers  $a_k \in [0, \infty)$ ,  $k \in \mathbb{N}$ , such that  $\sum_{k=1}^{\infty} a_k < \infty$  and such that for every  $\varepsilon \in (0, \infty)$  it holds that  $\sup_{k \in \mathbb{N}} (a_k \cdot k^\varepsilon) = \limsup_{k \rightarrow \infty} (a_k \cdot k^\varepsilon) = \infty$ .

*Remark 6.6.6.* Let  $f: \mathbb{R} \rightarrow \mathbb{C}$  be a continuous and 1-periodic function. In many cases it is difficult or impossible to compute  $\hat{f}(k)$ ,  $k \in \mathbb{Z}$  analytically. Instead one sometimes uses approximations of  $\hat{f}(k)$ ,  $k \in \mathbb{Z}$ . For instance, the *composite left rectangle rule* gives the approximation

$$\begin{aligned}\hat{f}(k) &= \int_0^1 f(t) w_{-k}(t) dt = \sum_{\ell=0}^{N-1} \int_{\frac{\ell}{N}}^{\frac{(\ell+1)}{N}} f(t) e^{-2\pi i k t} dt \\ &\approx \frac{1}{N} \left[ \sum_{\ell=0}^{N-1} f\left(\frac{\ell}{N}\right) e^{-2\pi i k \frac{\ell}{N}} \right] = \frac{1}{N} \left[ \sum_{\ell=0}^{N-1} f\left(\frac{\ell}{N}\right) \omega_N^{k\ell} \right]\end{aligned}\tag{6.6.10}$$

for  $k \in \mathbb{Z}$  and  $N \in \mathbb{N}$ . This motivates the next definition.

**Definition 6.6.10** (Approximations of Fourier coefficients). Let  $N \in \mathbb{N}$ . If  $f: [0, 1) \rightarrow \mathbb{C}$  is a function, then we define the function  $\widehat{f}_N: \mathbb{Z} \rightarrow \mathbb{C}$  by

$$\widehat{f}_N(k) := \frac{1}{N} \left[ \sum_{\ell=0}^{N-1} f\left(\frac{\ell}{N}\right) \omega_N^{k\ell} \right] \quad (6.6.11)$$

for all  $k \in \mathbb{Z}$ . In addition, if  $f: \mathbb{R} \rightarrow \mathbb{C}$  is a 1-periodic function, then we define the function  $\widehat{f}_N: \mathbb{Z} \rightarrow \mathbb{C}$  by  $\widehat{f}_N := \widehat{f|_{[0,1)N}}$ .

**Theorem 6.6.11** (Error of DFT; aliasing formula). Let  $N \in \mathbb{N}$  and let  $f: \mathbb{R} \rightarrow \mathbb{C}$  be a continuous and 1-periodic function with  $\sum_{k \in \mathbb{Z}} |\widehat{f}(k)| < \infty$ . Then

$$\forall k \in \mathbb{Z}: \quad \widehat{f}_N(k) - \widehat{f}(k) = \sum_{j \in \mathbb{Z} \setminus \{0\}} \widehat{f}(k + jN) .$$

*Proof of Theorem 6.6.11.* Note that Lemma 5.2.2 implies for every  $k \in \mathbb{Z}$  that

$$\begin{aligned}
 \hat{f}_N(k) &= \frac{1}{N} \left[ \sum_{\ell=0}^{N-1} f\left(\frac{\ell}{N}\right) \omega_N^{k\ell} \right] = \frac{1}{N} \left[ \sum_{\ell=0}^{N-1} \left[ \sum_{j=-\infty}^{\infty} \hat{f}(j) w_j\left(\frac{\ell}{N}\right) \right] \omega_N^{k\ell} \right] \\
 &= \frac{1}{N} \left[ \sum_{j=-\infty}^{\infty} \hat{f}(j) \left[ \sum_{\ell=0}^{N-1} \omega_N^{(k-j)\ell} \right] \right] = \sum_{\substack{j \in \mathbb{Z}, \\ k-j \in \{0, N, -N, \dots\}}} \hat{f}(j) \\
 &= \sum_{j \in \mathbb{Z}} \hat{f}(k + jN).
 \end{aligned} \tag{6.6.12}$$

This completes the proof of Theorem 6.6.11. □

**Corollary 6.6.12.** Let  $n, N \in \mathbb{N}$  and let  $f: \mathbb{R} \rightarrow \mathbb{C}$  be an  $n$ -times continuously differentiable and 1-periodic function. Then

$$|\hat{f}_N(k) - \hat{f}(k)| \leq \frac{\|f^{(n)}\|_{L^2((0,1), \mathbb{C})}}{N^n}$$

for all  $k \in \mathbb{Z}$  with  $|k| \leq \frac{N}{2}$ .

*Remark 6.6.7.* Let  $n, N \in \mathbb{N}$  and let  $f: \mathbb{R} \rightarrow \mathbb{C}$  be an  $n$ -times continuously differentiable and 1-periodic function. Then  $\hat{f}_N(k)$ ,  $k \in \mathbb{Z}$ , is  $N$ -periodic while  $\hat{f}(k) \rightarrow 0$  quickly as  $k \rightarrow \infty$ .  
 $\hat{f}_N(k)$  is thus a bad approximation of  $\hat{f}(k)$  for  $k$  large ( $k \gtrsim N$ ) but in the sense of Corollary 6.6.12 a good approximation for  $|k| \leq \frac{N}{2}$ .

### 6.6.3 Trigonometric Interpolation

Note: trigonometric polynomials  $\subset$  1-periodic functions from  $\mathbb{R}$  to  $\mathbb{C}$ .

*(Equidistant) trigonometric polynomial interpolation problem (cf. Series 7, Exercise 4)*

Given  $k \in \mathbb{N}_0$  and values  $y_0, y_1, \dots, y_{2k-1}, y_{2k} \in \mathbb{C}$  find  $p \in \mathcal{T}_k$  such that

$$\forall m \in \{0, 1, \dots, 2k\}: \quad p\left(\frac{m}{2k+1}\right) = y_m .$$

**Theorem 6.6.14.** *The trigonometric polynomial interpolation problem has a unique solution  $p \in \mathcal{T}_k$  given by*

$$p = \sum_{j=-k}^k \left[ \frac{1}{(2k+1)} \sum_{l=0}^{2k} y_l e^{\frac{-2\pi i j l}{(2k+1)}} \right] w_j = \sum_{j=-k}^k \left[ \frac{1}{(2k+1)} \sum_{l=0}^{2k} y_l \omega_{(2k+1)}^{jl} \right] w_j . \quad (6.6.13)$$

Reformulation of the trigonometric polynomial interpolation problem:

Let  $k \in \mathbb{N}_0$ , let  $\gamma_{-k}, \gamma_{1-k}, \dots, \gamma_k \in \mathbb{C}$ , let  $\mathbf{x} = (x_0, x_1, \dots, x_{2k}) \in \mathbb{C}^{2k+1}$  satisfy

$\forall j \in \{0, 1, \dots, k\}: \quad x_j = \gamma_{-j} \quad \text{and} \quad \forall j \in \{k+1, k+2, \dots, 2k\}: \quad x_j = \gamma_{2k+1-j} ,$

let  $\mathbf{y} = (y_0, y_1, \dots, y_{2k}) \in \mathbb{C}^{2k+1}$  and let  $p = \sum_{j=-k}^k \gamma_j w_j \in \mathcal{T}_k$ .

Then note that for all  $m \in \{0, 1, \dots, 2k\}$  that

$$\begin{aligned}
 p\left(\frac{m}{2k+1}\right) &= \sum_{j=-k}^k \gamma_j w_j\left(\frac{m}{2k+1}\right) = \sum_{j=-k}^k \gamma_j \omega_{(2k+1)}^{-jm} = \sum_{j=-k}^0 \gamma_j \omega_{(2k+1)}^{-jm} + \sum_{j=1}^k \gamma_j \omega_{(2k+1)}^{-jm} \\
 &= \sum_{j=0}^k \gamma_{-j} \omega_{(2k+1)}^{jm} + \sum_{j=1}^k \gamma_j \omega_{(2k+1)}^{(2k+1-j)m} = \sum_{j=0}^k \gamma_{-j} \omega_{(2k+1)}^{jm} + \sum_{j=k+1}^{2k} \gamma_{2k+1-j} \omega_{(2k+1)}^{jm} \\
 &= \sum_{j=0}^{2k} x_j \omega_{(2k+1)}^{jm} \quad \text{and this implies that } \left(p\left(\frac{0}{2k+1}\right), p\left(\frac{1}{2k+1}\right), \dots, p\left(\frac{2k}{2k+1}\right)\right) = \mathbf{F}_{2k+1} \mathbf{x}. \tag{6.6.14}
 \end{aligned}$$

Hence, the interpolation condition  $\forall m \in \{0, 1, \dots, 2k\}: p\left(\frac{m}{2k+1}\right) = y_m$  is equivalent to  $\mathbf{F}_{2k+1} \mathbf{x} = \mathbf{y}$  and this is equivalent to  $\mathbf{x} = \mathbf{F}_{2k+1}^{-1} \mathbf{y}$ , i.e.,

$$\forall j \in \{0, 1, \dots, 2k\}: \quad x_j = \frac{1}{(2k+1)} \sum_{m=0}^{2k} y_m \omega_{(2k+1)}^{-jm} \tag{6.6.15}$$

Condition (6.6.15) is in turn equivalent to

$$\forall j \in \{-k, 1-k, \dots, k\}: \quad \gamma_j = \frac{1}{(2k+1)} \sum_{m=0}^{2k} y_m \omega_{(2k+1)}^{jm} \tag{6.6.16}$$

The identity (6.6.14) can also be used to compute evaluations at equidistant nodes  $\frac{l}{(2k+1)}$ ,  $l \in \{0, 1, \dots, 2k\}$  of the trigonometric polynomial  $p \in \mathcal{T}_k$ .

**Definition 6.6.15** (Trigonometric interpolation operator). Let  $k \in \mathbb{N}_0$ . Then we denote by  $\mathsf{T}_k : \{f \in C(\mathbb{R}, \mathbb{C}) : f \text{ is 1-periodic}\} \rightarrow \mathcal{T}_k$  the linear mapping given by

$$\forall f \in \{f \in C(\mathbb{R}, \mathbb{C}) : f \text{ is 1-periodic}\} : \forall m \in \{0, 1, \dots, 2k\} : (\mathsf{T}_k(f))\left(\frac{m}{2k+1}\right) = f\left(\frac{m}{2k+1}\right).$$

**Corollary 6.6.16** (Trigonometric interpolation). Let  $k \in \mathbb{N}_0$  and let  $f : \mathbb{R} \rightarrow \mathbb{C}$  be a continuous and 1-periodic function. Then

$$\mathsf{T}_k(f) = \sum_{j=-k}^k \left[ \frac{1}{(2k+1)} \sum_{l=0}^{2k} f\left(\frac{l}{2k+1}\right) \omega_{(2k+1)}^{jl} \right] w_j = \sum_{j=-k}^k \widehat{f}_{2k+1}(j) w_j.$$

## 6.6.4 Trigonometric Interpolation: Error Estimates

The next corollary follows immediately from Theorem 6.6.14.

**Corollary 6.6.17** (Sampling-Theorem). *Let  $k \in \mathbb{N}_0$ . Then  $\mathbf{T}_k(f) = f$  for all  $f \in \mathcal{T}_k$ .*



data compression

**Theorem 6.6.18** (Error estimate for trigonometric interpolation). *Let  $k, n \in \mathbb{N}$ , let  $f: \mathbb{R} \rightarrow \mathbb{C}$  be an  $n$ -times continuously differentiable and 1-periodic function. Then*

$$\|f - \mathbf{T}_k(f)\|_{L^2((0,1),\mathbb{C})} \leq \frac{\|f^{(n)}\|_{L^2((0,1),\mathbb{C})}}{k^n}.$$

*Proof of Theorem 6.6.18.* Define the function  $e: \mathbb{R} \rightarrow \mathbb{C}$  through  $e(t) := f(t) - (\mathsf{T}_k(f))(t)$  for all  $t \in \mathbb{R}$ . Then note that

$$f = \sum_{j=-\infty}^{\infty} \widehat{f}(j) w_j, \quad \mathsf{T}_k(f) = \sum_{j=-k}^k \widehat{f}_{2k+1}(j) w_j, \quad e = \sum_{j=-\infty}^{\infty} \widehat{e}(j) w_j \quad (6.6.17)$$

and

$$\forall j \in \{-k, 1-k, \dots, k\}: \quad \widehat{e}(j) = \begin{cases} \widehat{f}(j) - \widehat{f}_{2k+1}(j) & : -k \leq j \leq k \\ \widehat{f}(j) & : |j| > k \end{cases}. \quad (6.6.18)$$

Parseval's identity (Lemma 6.6.7) and Theorem 6.6.11 hence show that

$$\begin{aligned} \|f - \mathsf{T}_k(f)\|_{L^2((0,1), \mathbb{C})}^2 &= \|e\|_{L^2((0,1), \mathbb{C})}^2 = \sum_{j \in \mathbb{Z}} |\widehat{e}(j)|^2 \\ &= \sum_{j=-k}^k |\widehat{f}(j) - \widehat{f}_{2k+1}(j)|^2 + \sum_{\substack{j \in \mathbb{Z} \\ |j| > k}} |\widehat{f}(j)|^2 \end{aligned} \quad (6.6.19)$$

$$\leq \sum_{j=-k}^k \left| \sum_{l \in \mathbb{Z} \setminus \{0\}} \widehat{f}(j + l(2k+1)) \right|^2 + \frac{1}{k^{2n}} \left[ \sum_{\substack{j \in \mathbb{Z} \\ |j| > k}} j^{2n} |\widehat{f}(j)|^2 \right]$$

and this together with the Cauchy-Schwarz inequality implies that

$$\begin{aligned} & \|f - \mathcal{T}_k(f)\|_{L^2((0,1),\mathbb{C})}^2 \\ & \leq \sum_{j=-k}^k \left[ \sum_{l \in \mathbb{Z} \setminus \{0\}} |\hat{f}(j + l(2k + 1))|^2 [j + l(2k + 1)]^{2n} \right] \left[ \sum_{l \in \mathbb{Z} \setminus \{0\}} \frac{1}{[j + l(2k + 1)]^{2n}} \right] \\ & \quad + \frac{\|f^{(n)}\|_{L^2((0,1),\mathbb{C})}^2}{(2\pi k)^{2n}}. \end{aligned} \quad (6.6.20)$$

Hence, we obtain

$$\begin{aligned} & \|f - \mathcal{T}_k(f)\|_{L^2((0,1),\mathbb{C})}^2 \leq \frac{\|f^{(n)}\|_{L^2((0,1),\mathbb{C})}^2}{(2\pi k)^{2n}} \\ & \quad + \left[ \sum_{j \in \mathbb{Z}} |\hat{f}(j)|^2 |j|^{2n} \right] \left[ \sum_{l \in \mathbb{Z} \setminus \{0\}} \frac{1}{[(|l| - 1/2)(2k + 1)]^{2n}} \right] \end{aligned} \quad (6.6.21)$$

and this proves that

$$\|f - \mathcal{T}_k(f)\|_{L^2((0,1),\mathbb{C})}^2 \leq \frac{\|f^{(n)}\|_{L^2((0,1),\mathbb{C})}^2}{(2\pi k)^{2n}} + \frac{\|f^{(n)}\|_{L^2((0,1),\mathbb{C})}^2}{(2\pi)^{2n} (2k + 1)^{2n}} \left[ \sum_{l=0}^{\infty} \frac{2}{(l + 1/2)^{2n}} \right]. \quad (6.6.22)$$

This finally shows that

$$\begin{aligned}\|f - T_k(f)\|_{L^2((0,1), \mathbb{C})}^2 &\leq \frac{\|f^{(n)}\|_{L^2((0,1), \mathbb{C})}^2}{(2\pi k)^{2n}} \left[ 1 + \sum_{l=0}^{\infty} \frac{2}{(2l+1)^{2n}} \right] \\ &\leq \frac{\|f^{(n)}\|_{L^2((0,1), \mathbb{C})}^2}{k^{2n}}.\end{aligned}\tag{6.6.23}$$

The proof of Theorem 6.6.18 is thus completed. □