Algebraic Geometry

D-MATH, FS 2014

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Exercise Sheet 5

1. Let $G(2,4)$ be the Grassmannian of 2-planes in $\mathbb{C}^4$ and let

$$\gamma : G(2,4) \longrightarrow \mathbb{P}(\bigwedge^2 \mathbb{C}^4) \cong \mathbb{CP}^5$$

be the Plücker Embedding given by $\gamma(\Lambda) = [v_1 \wedge v_2]$, where $\Lambda \in G(2,4)$ and $v_1, v_2 \in \mathbb{C}^4$ is a basis of Λ.

We say, a vector $\omega \in \bigwedge^2 \mathbb{C}^4$ is decomposable if there exist v_1, v_2 such that $\omega = v_1 \wedge v_2$. Let $X \subset \mathbb{P}(\bigwedge^2 \mathbb{C}^4)$ be the locus of decomposable vectors.

a) Show that $\omega \in \bigwedge^2 \mathbb{C}^4$ is decomposable if and only if

$$s(\omega) : \mathbb{C}^4 \longrightarrow \bigwedge^3 \mathbb{C}^4, \qquad v \longrightarrow v \wedge \omega$$

has a two-dimensional kernel.

b) Prove that γ is a bijection of $G(2,4)$ onto X. (Hint: Define $f : X \longrightarrow G(2,4)$ by $f([\omega]) = \{v \in \mathbb{C}^4 \mid v \wedge \omega = 0\}$ and show that this defines an inverse to γ.)

c) Show that ω is decomposable if and only if $\omega \wedge \omega = 0$.

d) Let $e_i, i = 1, \ldots, 4$ be the standard basis on $\mathbb{C}^4$. Let $e_i \wedge e_j, 1 \leq i < j \leq 4$ be the induced basis for $\bigwedge^2 \mathbb{C}^4$. Writing each element $\omega \in \bigwedge^2 \mathbb{C}^4$ as $\omega = \sum_{i<j} a_{ij} e_i \wedge e_j$ induces homogeneous coordinates $[(a_{ij})_{i<j}]$ on $\mathbb{P}(\bigwedge^2 \mathbb{C}^4)$. Use *c*) to show that

$$X = V(a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23})$$

This exercise shows that X is a projective variety and by giving $G(2,4)$ the induced structure as a projective variety, we conclude that $G(2,4)$ is via the Plücker embedding isomorphic to a quadric hypersurface in $\mathbb{CP}^5$.

2. Let $f : X \longrightarrow Y$ be an algebraic map between quasi-projective varieties X and Y. Let $Z \subset Y$ be a closed subvariety and suppose that $\mathrm{Im}(f)$ lies in Z. Prove that the induced map $f : X \longrightarrow Z$ is algebraic.

3. Consider the category of

(*) affine algebraic varieties with algebraic maps,

(**) finitely generated reduced $\mathbb{C}$-algebras with $\mathbb{C}$-algebra homomorphism.

Prove the equivalence of the categories (*) and (**) via the contravariant functor

$$V \longrightarrow \Gamma(V)$$

4. For every postitive integer n, let L_n be the line in $\mathbb{CP}^3$ defined by

$$\left\{ [x, y, nx, ny] \;\middle|\; [x, y] \in \mathbb{CP}^1 \right\}$$

Find the equations defining the Zariski closure of the infinite union

$$L_1 \cup L_2 \cup L_3 \cup \ldots$$

5. Let X, Y, Z, W be quasi projective varieties with algebraic maps $f : X \longrightarrow Z$ and $g : Y \longrightarrow W$. Prove that

$$(f, g) : X \times Y \longrightarrow Z \times W$$

is an algebraic map.

6. Let $X = \mathbb{CP}^2 \setminus \{\text{point}\}$. Is there a non-constant algebraic map $f : X \longrightarrow \mathbb{C}$?

Due on March 28, 2014