Algebraic Geometry

D-MATH, FS 2014

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Solutions Sheet 1

1./2. Any Algebraic Geometry textbook.

3. This can be found in a slightly different language in Hartshorne Proposition II.2.2.

4. Cover $\mathbb{CP}^n$ with the standard covering $U_i = \{x_i \neq 0\}$ and recall that U_i is isomorphic to $\mathbb{C}^n$ with the coordinates $\frac{x_0}{x_i}, \dots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \dots, \frac{x_n}{x_i}$. Hence given an algebraic function f on $\mathbb{CP}^n$, we know that $f_{|U_i}$ is a polynomial f_i in $\frac{x_0}{x_i}, \dots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \dots, \frac{x_n}{x_i}$ (e.g. by problem 3).

Now consider two open subsets, lets say, U_0 and U_1. On the intersection we must have

$$f_0 = \sum_I a_I \left(\frac{x_1}{x_0}\right)^{i_1} \dots \left(\frac{x_n}{x_0}\right)^{i_n} = \sum_J b_J \left(\frac{x_0}{x_1}\right)^{j_1} \left(\frac{x_2}{x_1}\right)^{j_2} \dots \left(\frac{x_n}{x_1}\right)^{i_n} = f_1$$

The left hand side has only terms with positive x_i exponents, while the right hand side has only negative ones. This shows $a_I = 0$ except for the constant term. Therefore $f_0 = f_{|U_0} = \text{const}$. But with the same argument also $f_i = \text{const}$ for all i. As they agree on the overlap, we are done.

5. Let x, y be the two coordinates of $\mathbb{C}^2$. Let $D(x) = \{p \in \mathbb{C}^2 \mid x(p) \neq 0\}$, i.e. with nonvanishing first coordinate. Let us first calculate the regular functions on $D(x)$. As an variety $D(x)$ is isomorphic to $Y = V(xz-1) \subset \mathbb{C}^3$ and so the ring of regular function on $D(x)$ is given by $\mathbb{C}[x,y,z]/(xz-1) = \mathbb{C}[x,y,\frac{1}{x}]$. Similar statements apply for $D(y)$ and $D(xy) = D(x) \cap D(y)$.

Now consider $X = \mathbb{C}^2 \setminus \{(0,0)\}$ and note that $X = D(x) \cup D(y)$. We can identify regular functions on X with regular functions on $D(x)$ and $D(y)$ that coincide on their common domain of definition, namely $D(xy) = D(x) \cap D(y)$. As the restriction map $\mathbb{C}[x,y,\frac{1}{x}] \longrightarrow \mathbb{C}[x,y,\frac{1}{xy}] = \mathbb{C}[x,y,\frac{1}{x},\frac{1}{y}]$ is an injection, we have reduced the problem to calculating the intersection $\mathbb{C}[x,y,\frac{1}{x}] \cap \mathbb{C}[x,y,\frac{1}{y}]$ considered as subrings of $\mathbb{C}[x,y,\frac{1}{x},\frac{1}{y}]$. Given such a g, we see that its denominator must be a constant and so the intersection is just $\mathbb{C}[x,y]$. Therefore we have found that the ring of regular functions on X is just $\mathbb{C}[x,y]$.