Solutions to problem set 1

1. Consider the diagram

$$\begin{tikzcd} H_n(\Delta^n, \partial\Delta^n) \arrow[r, "\cong"] \arrow[d, "\cong"] & \widetilde{H}_{n-1}(\partial\Delta^n) \\ H_n(\Delta^n/\partial\Delta^n, *) & \end{tikzcd} \tag{1}$$

The horizontal map is the boundary map from the (reduced) LES for the pair $(\Delta^n, \partial\Delta^n)$, which is an isomorphism by looking at the neighbouring terms in the LES. The vertical map is induced by the quotient map $(\Delta^n, \partial\Delta^n) \to (\Delta^n/\partial\Delta^n, *)$ and is an isomorphism since $(\Delta^n, \partial\Delta^n)$ is a good pair.

Consider now the tautological n-simplex $\alpha_n : \Delta^n \to \Delta^n$, which defines a class $[\alpha_n] \in H_n(\Delta^n, \partial\Delta^n)$. The image of $[\alpha_n]$ under the vertical map is $[\sigma_n] \in H_n(\Delta^n/\partial\Delta^n, *)$, while its image under the horizontal map is the class $[\beta_{n-1}] \in \widetilde{H}_{n-1}(\partial\Delta^n)$ with

$$\beta_{n-1} = \partial_n \alpha_n = \sum_{i=0}^{n} (-1)^i F_i^n \in C_{n-1}(\partial\Delta^n),$$

where $F_i^n : \Delta^{n-1} \to \partial\Delta^n$ is the i-th face map of the simplex Δ^n. So once we know that $[\beta_{n-1}]$ generates $\widetilde{H}_{n-1}(\partial\Delta^n)$, we can conclude from (1) that $[\sigma_n]$ generates $H_n(\Delta^n/\partial\Delta^n, *)$.

It is clear that $[\beta_0]$ generates $\widetilde{H}_0(\partial\Delta^1)$, so we know that $[\sigma_1]$ generates $H_1(\Delta^1/\partial\Delta^1, *)$, which is what the problem asks us to prove for $n = 1$. We now proceed by induction; for the inductive step, consider the map $\phi : \partial\Delta^n \to \Delta^{n-1}/\partial\Delta^{n-1}$ which collapses all except the zero-th face to a point, and the induced map $\phi_* : H_{n-1}(\partial\Delta^n) \to H_{n-1}(\Delta^{n-1}/\partial\Delta^{n-1}, *)$. Observe that $\phi_*[\beta_{n-1}] = [\sigma_{n-1}]$; since $[\sigma_{n-1}]$ generates by inductive assumption, we conclude that $[\beta_{n-1}]$ generates.

2. Consider the cover of Y given by the subsets $A = \Delta^n_+$ and $B = \Delta^n_-$. Both are contractible and we have $A \cap B = \partial\Delta^n$, so that the relevant piece of the corresponding reduced MV sequence reads

$$0 \to \widetilde{H}_n(Y) \xrightarrow{\partial_*} \widetilde{H}_{n-1}(\partial\Delta^n) \to 0$$

Note that $\partial_*[\tau_+ - \tau_-] = [\partial\tau_+] = [\beta_{n-1}] \in \widetilde{H}_{n-1}(\partial\Delta^n)$ with $\beta_{n-1} \in C_{n-1}(\partial\Delta^n)$ defined as in the solution to the previous problem. Since $[\beta_{n-1}]$ generates (see the previous problem) we deduce that $[\tau_+ - \tau_-]$ generates.

We give an alternative inductive proof that $[\beta_n]$ generates $\widetilde{H}_{n-1}(\partial\Delta^n)$ using the Mayer-Vietoris sequence. For $n = 0$ the statement is clear. For the inductive step, consider the cover of $\partial\Delta^{n+1}$ given by $A := \operatorname{im} F_0^{n+1}$ and $B := \partial\Delta^{n+1} \setminus \operatorname{int} A$ (the interiors don't cover all of $\partial\Delta^{n+1}$, but that can be repaired by taking small thickenings of A and B). Since both A and B are contractible, the corresponding reduced MV sequence splits into pieces of the form

$$0 \to \widetilde{H}_n(\partial\Delta^{n+1}) \xrightarrow{\cong} \widetilde{H}_{n-1}(A \cap B) \to 0$$

Note that we can identify $A \cap B = \partial A$ with $\partial\Delta^n$ via $F_0^n|_{\partial\Delta^n}$. By definition of the MV boundary map $\partial_* : \widetilde{H}_n(\partial\Delta^{n+1}) \to \widetilde{H}_{n-1}(A \cap B)$, we have $\partial_*[\beta_n] = [\partial F_0^{n+1}]$, which in our identification $A \cap B \cong \partial\Delta^n$ is $[\beta_{n-1}]$. Since ∂_* is an isomorphism and $[\beta_{n-1}]$ generates $\widetilde{H}_{n-1}(\partial\Delta^n)$ by inductive assumption, it follows that $[\beta_n]$ generates $\widetilde{H}_n(\partial\Delta^{n+1})$.

3. *In the following, all homology groups have $\mathbb{Z}_2$ coefficients.* Given that $H_k(\mathbb{R}P^n) = 0$ for $k > n$ by assumption, the leftmost piece of the Smith sequence for the cover $p : S^n \to \mathbb{R}P^n$ looks like

$$0 \to H_n(\mathbb{R}P^n) \xrightarrow{t_*} H_n(S^n) \xrightarrow{p_*} H_n(\mathbb{R}P^n) \xrightarrow{\partial_*} H_{n-1}(\mathbb{R}P^n) \to H_{n-1}(S^n) = 0 \to \dots$$

Here t_* is induced by the map $C_*(\mathbb{R}P^n) \to C_*(S^n)$ taking a simplex $\sigma : \Delta^k \to \mathbb{R}P^k$ to $\tilde{\sigma} + \alpha \circ \tilde{\sigma}$, where $\tilde{\sigma} : \Delta^n \to S^n$ is one of the two possible lifts of σ to S^n and where $\alpha : S^n \to S^n$ denotes the antipodal map. Note that we have $t_* \circ p_* = (\mathrm{id} + \alpha_*) : H_*(S^n) \to H_*(S^n)$, which implies $t_* \circ p_* = 0$ because $\alpha_* = \mathrm{id} : H_*(S^n) \to H_*(S^n)$ (because α_* is an involution and $H_k(S^n)$ either vanishes or is $\mathbb{Z}_2$). This together with the fact that $t_* : H_n(\mathbb{R}P^n) \to H_n(S^n)$ is injective implies that $p_* : H_n(S^n) \to H_n(\mathbb{R}P^n)$ vanishes, and hence $t_* : H_n(\mathbb{R}P^n) \to H_n(S^n) \cong \mathbb{Z}_2$ is an isomorphism. Moreover, $p_* = 0$ implies that $\partial_* : H_n(\mathbb{R}P^n) \to H_{n-1}(\mathbb{R}P^n)$ is an isomorphism, and the same is true for $\partial : H_k(\mathbb{R}P^n) \to H_{k-1}(\mathbb{R}P^n)$ for $k > 0$ since $H_*(S^n) = 0$ except in degrees 0 and n. Inductively we obtain $H_k(\mathbb{R}P^n) \cong \mathbb{Z}_2$ for all $0 \leq k \leq n$.

4. We can lift the composition $f \circ p : S^n \to \mathbb{R}P^n \to \mathbb{R}P^m$ to a map $\tilde{f} : S^n \to S^m$ because $\pi_1(S^n)$ is trivial as $n > 1$ by assumption. The induced maps $f_*, \tilde{f}_*$ give a map from the Smith sequence for $S^n \to \mathbb{R}P^n$ to that for $S^m \to \mathbb{R}P^m$ in which all squares commute. The leftmost nontrivial square is

$$\begin{array}{ccc} H_m(\mathbb{R}P^n) & \longrightarrow & H_m(S^n) \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle \tilde{f}_*} \\ H_m(\mathbb{R}P^m) & \xrightarrow{\cong} & H_m(S^m) \end{array}$$

The lower map is an isomorphism by the reasoning in the solution for the previous problem; since $H_m(S^n) = 0$, it follows that $f_* : H_m(\mathbb{R}P^n) \to H_m(\mathbb{R}P^m)$ vanishes. Using the squares

$$\begin{array}{ccc} H_k(\mathbb{R}P^n) & \xrightarrow{\partial_*} & H_{k-1}(\mathbb{R}P^n) \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle f_*} \\ H_k(\mathbb{R}P^m) & \xrightarrow{\partial_*} & H_{k-1}(\mathbb{R}P^m) \end{array}$$

and the fact that the boundary maps are isomorphism, we obtain inductively that $f_* : H_k(\mathbb{R}P^n) \to H_k(\mathbb{R}P^m)$ vanishes in all degrees $k \geq 1$. In particular, $f_* : H_1(\mathbb{R}P^n) \to H_1(\mathbb{R}P^n)$ vanishes, which implies that $f_\# : \pi_1(\mathbb{R}P^n) \to \pi_1(\mathbb{R}P^m)$ vanishes as the natural maps $\pi_1(\mathbb{R}P^i) \to H_1(\mathbb{R}P^i)$ are isomorphisms.

5. Assume that $r : \mathbb{R}P^3 \to \mathbb{R}P^2$ is a retraction and denote by $i : \mathbb{R}P^2 \hookrightarrow \mathbb{R}P^3$ the inclusion. Then we have $r \circ i = \mathrm{id}_{\mathbb{R}P^2}$ and hence $(r \circ i)_\# = \mathrm{id} : \pi_1(\mathbb{R}P^2) \to \pi_1(\mathbb{R}P^2)$, which is non-zero because $\pi_1(\mathbb{R}P^2) = H_1(\mathbb{R}P^2; \mathbb{Z}) = \mathbb{Z}_2$. On the other hand, we have $(r \circ i)_\# = r_\# \circ i_\# = 0$ since $r_\# = 0$ by the previous exercise. That is a contradiction.

6. Cf. the proof of Borsuk-Ulam in [Hatcher, pp. 174-176]!

7. Let $n > m$ and supposed that there exists an equivariant map $\phi : S^n \to S^m$, i.e., such that $\phi(-x) = -\phi(x)$ for all x. Consider the map $f : S^{m+1} \to \mathbb{R}^{m+1}$ obtained by composing the restriction of ϕ to $S^{m+1} \subseteq S^n$ with the inclusion $S^m \hookrightarrow \mathbb{R}^{m+1}$. This map satisfies $f(-x) = -f(x)$ for all $x \in S^{m+1}$. Since $f(x) \in S^{m+1}$ and hence $f(x) \neq -f(x)$, we conclude $f(-x) \neq f(x)$ for all $x \in S^{m+1}$, which contradicts the Borsuk-Ulam theorem.

8. Cf. [Bredon, Corollary IV.20.4]!