D-MATH
Prof. Brent Doran

Algebra II

FS 2014

Serie 1

Quotient rings, adjoining elements and product rings

1. Consider the homomorphism $\mathbb{Z}[x] \to \mathbb{Z}$ for which $x \mapsto 1$. Explain in this case what the Correspondence Theorem says about ideals of $\mathbb{Z}[x]$.

2. (a) Let $\mathfrak{I} \subset \mathbb{Z}[x]$ be the ideal generated by $x - 3$ and 7. Show that for every $f(x) \in \mathbb{Z}[x]$, there is an integer $0 \leqslant \alpha \leqslant 6$ such that $f(x) - \alpha \in \mathfrak{I}$. Conclude that the quotient ring $\mathbb{Z}[x]/\mathfrak{I}$ is isomorphic to $\mathbb{Z}/(7)$.
   (b) Find α explicitly for $f(x) = x^{250} + 15x^{14} + x^2 + 5$. (**Hint** : you may want to use here Fermat's Little Theorem, see e.g. `http://www.math.ethz.ch/education/bachelor/lectures/hs2013/math/algebra1/Exercise6.pdf`).
   (c) Describe the ring obtained from $\mathbb{Z}/(12)$ by adjoining an inverse of 2.

3. Let $R = K[t]$ be a polynomial ring over a field K and consider the ring $R' = R[x]/(tx - 1)$ obtained by adjoining an inverse of t to R. Prove that R' can be identified as the ring of **Laurent polynomials**.

4. Let $\mathfrak{I}$ and $\mathfrak{J}$ be ideals of a ring R such that $\mathfrak{I} + \mathfrak{J} = R$. Prove :
   (a) $\mathfrak{I}\mathfrak{J} = \mathfrak{I} \cap \mathfrak{J}$.
   (b) (the **Chinese Remainder Theorem**) For any $a, b \in R$, there is an element $x \in R$ such that $x \equiv a \bmod \mathfrak{I}$ and $x \equiv b \bmod \mathfrak{J}$.
   (c) If $\mathfrak{I}\mathfrak{J} = 0$, then R is isomorphic to the product ring $R/\mathfrak{I} \times R/\mathfrak{J}$.
   (d) Describe the idempotent elements corresponding to the above product decomposition.

5. Andy, Esther and Nick are flatmates in a WOKO and want to have pizza all together one night. However, they all have their quirks : Andy eats pizza every fifth day, Esther every 7th and Nick every 11th. Given that in 2014, Nick and Andy had their first pizza together on January 3 and Esther had pizza on January 4, on what day(s) of 2014 will they all manage to have pizza together ?

6. Is $\mathbb{Z}/(6)$ isomorphic as a ring to the product $\mathbb{Z}/(2) \times \mathbb{Z}/(3)$? What about $\mathbb{Z}/(8)$ and $\mathbb{Z}/(2) \times \mathbb{Z}/(4)$?