Exercise sheet 11

Galois groups

1. Show that the polynomials $(x^2-2x-2)(x^2+1)$ and $x^5-3x^3+x^2-3$ have the same splitting field over $\mathbb{Q}$. What is the degree of the field extension ?

2. Let F be a field. Show that the field extension $F(x)/F$ admits a F-endomorphism of $F(x)$ that is *not* an automorphism.

3. Let $f(x)$ be an irreducible polynomial over a field F and denote by K its splitting field. Prove that if the Galois group $G = \mathrm{Gal}(K/F)$ is abelian, then $K = F(\alpha)$ for any root α of $f(x)$.

4. Exhibit a polynomial $f(x) \in \mathbb{Q}[x]$ of even degree $n \geqslant 2$ with Galois group $\mathbb{Z}/(2)$.

5. Consider the group

$$H = \left\{ \sigma_a : a \in \mathbb{C},\ \sigma_a\left(\frac{g(x)}{h(x)}\right) = \frac{g(x+a)}{h(x+a)} \right\}$$

of $\mathbb{C}$-automorphisms of the field $\mathbb{C}(t)$ of rational functions. Show that $\mathbb{C}(t)^H = \mathbb{C}$.