Exercise sheet 12

Galois extensions and Galois correspondence

1. Consider the polynomial $f(x) = x^2 - 2$. Determine the Galois group of $K/\mathbb{Q}$, where K is the splitting field. The same question as above for

$$g(x) = (x^2 - 2)(x^2 - 3).$$

Then, via the Galois correspondence, give the factorisation of g over each intermediate field $\mathbb{Q} \subset L \subset K$.

2. Let $q = p^n$ be the n-th power of a prime p. Show that the extension $\mathbb{F}_q/\mathbb{F}_p$ is Galois and that its Galois group is the cyclic group C_n generated by the Frobenius endomorphism $\Phi_p(x) = x^p$. Prove that the Main Theorem of Galois theory is true for this extension.

3. Set $K = \mathbb{Q}(\sqrt[3]{2}, \omega)$ for $\omega = e^{2\pi i/3}$. Show that $K/\mathbb{Q}$ is Galois and that its Galois group is isomorphic to S_3. Describe the Galois correspondence for this particular example.

4. We give a proof of the Fundamental Theorem of Algebra using Galois theory. Let K be a finite field extension of $\mathbb{R}$.

   (a) Assume that $K/\mathbb{R}$ is a Galois extension. Show that there is a chain of fields

   $$\mathbb{R} \subset K_1 \subset \cdots \subset K_n = K$$

   such that

   i. $[K_{i+1} : K_i] = 2$, for $1 \leqslant i \leqslant n-1$,

   ii. $[K_1 : \mathbb{R}]$ odd.

   (b) Show that if $[K : \mathbb{R}] = 2$, then K is isomorphic to $\mathbb{C}$.

   (c) Show that if $[K : \mathbb{R}]$ is odd, then $K = \mathbb{R}$.

   (d) Conclude that K is either $\mathbb{R}$ or $\mathbb{C}$.

* **Hints :** *In exercise 3, you may consider the action of the Galois group on the set* $\{\sqrt[3]{2}, \sqrt[3]{2}\omega, \sqrt[3]{2}\omega^2\}$. *In exercise 4(a), recall the first Sylow theorem.*