D-MATH | Algebra II | FS 2014
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Serie 2

Integral domains, fraction fields and maximal ideals

1. (a) Is there an integral domain that contains exactly 15 elements ?

   (b) Let $\mathfrak{I} \subset R$ be an ideal. Prove that the quotient $R/\mathfrak{I}$ is an integral domain if and only if $\mathfrak{I}$ is a **prime ideal**. (An ideal $\mathfrak{I} \subsetneq R$ is prime if, for any two elements $a, b \in R$, $ab \in \mathfrak{I}$ then either $a \in \mathfrak{I}$ or $b \in \mathfrak{I}$.)

2. (a) Show that the quotient $R = \mathbb{Q}[x, y]/(x^2 + y^2 - 1)$ is an integral domain.

   (b) Write down the stereographic projection of the circle $x^2 + y^2 = 1$. Use it to show that the fraction field of R is isomorphic to the field of rational functions $\mathbb{Q}(t)$.

3. (a) Let $F[x]$ be a polynomial ring over a field F. Prove that the maximal ideals of $F[x]$ are the principal ideals generated by monic irreducible polynomials.

   (b) Which principal ideals of $\mathbb{Z}[x]$ are maximal ?

   (c) Find the maximal ideals of the following rings :

$$R_1 = \mathbb{R} \times \mathbb{R}, \quad R_2 = \mathbb{R}[x]/(x^2 - 3x + 2), \quad R_3 = \mathbb{R}[x]/(x^2 + x + 1).$$