Algebra II

Serie 3

Unique factorization domains

1. Show that $\mathbb{Z}[\sqrt{2}]$ is a Euclidean domain.
2. (a) Show that the size function on $\mathbb{Z}[i]$ is multiplicative.
   (b) Describe a systematic way to do division with remainder in $\mathbb{Z}[i]$, and use it to divide $4 + 36i$ by $5 + i$.
   (c) Let $a, b \in \mathbb{Z}$. Show that their greatest common divisors in $\mathbb{Z}$ and $\mathbb{Z}[i]$ coincide.
   (d) Let $p \in \mathbb{N}$ be a prime with $p \equiv 3 \mod 4$. Show that p is also prime in $\mathbb{Z}[i]$.
   (e) Decompose $-1 + 3i$ into irreducible factors in $\mathbb{Z}[i]$
3. Decompose $x^3 + x + 2$ into irreducible factors in $\mathbb{F}_3[x]$.
4. Let $F[x]$ be a polynomial ring over a field F. Prove that there are infinitely many monic irreducible polynomials in $F[x]$.
   Hint : Check out Euclid's proof of the infinitude of primes.
5. Establish a bijective correspondence between maximal ideals of $\mathbb{R}[x]$ and points in the upper half-plane $\{(x, y) : x, y \in \mathbb{R}, y \geqslant 0\}$.