# Problem set 5

1. Show that $\sqrt{7}$, $e^{2\pi i/17}$, $\sqrt{2}+\sqrt[3]{5}$ are algebraic integers over $\mathbb{Q}$. Show that all complex numbers are algebraic over $\mathbb{R}$.

2. Let $d, d'$ be square-free integers.

   (a) Show that $\mathrm{Aut}(\mathbb{Q}[\sqrt{d}])$ is a group of order two, that consists of the identity and the map $\sigma(a + b\sqrt{d}) = a - b\sqrt{d}$.

   (b) When are $\mathbb{Q}[\sqrt{d}]$ and $\mathbb{Q}[\sqrt{d'}]$ not isomorphic ? Conclude that there are countably many distinct quadratic number fields $\mathbb{Q}[\sqrt{d}]$.

   (c) Show that uncountably many transcendental numbers exist.

3. Determine the integer $d$ for which the polynomials

$$f(x) = x^5 - 8x^3 + 9x - 3, \qquad g(x) = x^4 - 5x^2 - 6x + 3$$

   have a common root in $\mathbb{Q}[\sqrt{d}]$.

4. For which negative integers $d \equiv 2 \bmod 4$ is the ring of integers in $\mathbb{Q}[\sqrt{d}]$ a unique factorisation domain ?