D-MATH | Algebra II | FS 2014

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Problem set 6

Field extensions, irreducible and transcendental elements

1. Let α be a complex root of $x^3 - 3x + 4$. Find the inverse of $\alpha^2 + \alpha + 1$ in the form $a\alpha^2 + b\alpha + c$, with $a, b, c \in \mathbb{Q}$.

2. Let $\beta = \sqrt[3]{2}e^{2\pi i/3}$. Prove that $x_1^2 + \cdots + x_k^2 = -1$, $k \geqslant 1$, has no solutions with all $x_i \in \mathbb{Q}(\beta)$.

3. (a) Show that $\sqrt{3} \notin \mathbb{Q}$, and $\sqrt{2} \notin \mathbb{Q}(\sqrt{3})$.

   (b) Show that $\mathbb{Q}(\sqrt{2}, \sqrt{3}) = \mathbb{Q}(\sqrt{2} + \sqrt{3})$.

   (c) Determine the degrees of the extensions $\mathbb{Q}(\sqrt{3})$ over $\mathbb{Q}$ and $\mathbb{Q}(\sqrt{2}, \sqrt{3})$ over $\mathbb{Q}(\sqrt{3})$.

4. Let $K = F(\alpha)$ be a field extension generated by a transcendental element α, and let β be an element of K that is not in F. Prove that α is algebraic over the field $F(\beta)$.