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Problem set 7

Degree of field extension, minimal polynomial

1. (a) Let F be a field, and let α be an element that generates a field extension of F of degree 5. Prove that α^2 generates the same extension.

   (b) Prove the last statement for 5 replaced by any odd integer.

2. Prove that $x^4 + 3x + 3$ is irreducible over $\mathbb{Q}[\sqrt[3]{2}]$.

3. Let $K = \mathbb{Q}(\alpha)$ where α is a root of $x^3 - x - 1$. Determine the irreducible polynomial for $1 + \alpha^2$ over $\mathbb{Q}$.

4. Determine the irreducible polynomials for $\alpha = \sqrt{3} + \sqrt{5}$ over the following fields

$$\mathbb{Q}, \quad \mathbb{Q}[\sqrt{5}], \quad \mathbb{Q}[\sqrt{10}], \quad \mathbb{Q}[\sqrt{15}].$$

5. A field extension K/F is an algebraic extension if every element of K is algebraic over F.

   (a) Let L/K and K/F be algebraic extensions. Prove that L/F is an algebraic extension.

   (b) Let $\alpha, \beta \in \mathbb{C}$. Prove that if $\alpha + \beta$ and $\alpha\beta$ are algebraic numbers, then α and β are also algebraic numbers.