D-MATH Algebra II FS 2014
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Solutions 10

Relating roots and coefficients of a polynomial : elementary symmetric polynomials and discriminant

1. Solve the following system in $\mathbb{C}$

$$\begin{cases} z_1 + z_2 + z_3 & = 1 \\ z_1 z_2 z_3 & = 1 \\ |z_1| = |z_2| = |z_3| & = 1. \end{cases}$$

Solution : Let z_1, z_2, z_3 be solutions of the system above. We determine the coefficients of the polynomial

$$f(x) = (x - z_1)(x - z_2)(x - z_3).$$

The associated elementary symmetric functions are

$$s_1 = z_1 + z_2 + z_3 = 1$$

$$s_2 = z_1 z_2 + z_1 z_3 + z_2 z_3 = \frac{1}{z_3} + \frac{1}{z_2} + \frac{1}{z_1} = \overline{z_1} + \overline{z_2} + \overline{z_3} = 1$$

$$s_3 = z_1 z_2 z_3 = 1$$

so we can express

$$\begin{aligned} f(x) &= x^3 - x^2 + x - 1 \\ &= (x-1)(x^2+1) = (x-1)(x+i)(x-i). \end{aligned}$$

2. Consider $f(x) = x^3 - 2x + 5$ and denote $\alpha_1, \alpha_2, \alpha_3$ its complex roots.

   (a) Compute $\alpha_1^4 + \alpha_2^4 + \alpha_3^4$.
   Solution : We can read off the elementary symmetric functions from $f(x)$:

$$s_1 = 0, \quad s_2 = -2, \quad s_3 = -5.$$

   For each α_i, $\alpha_i^4 = 2\alpha_i^2 - 5\alpha_i$ holds, hence

$$\sum \alpha_i^4 = 2 \sum \alpha_i^2 - 5 \sum \alpha_i = 2\left(s_1^2 - 2s_2\right) - 5s_1 = 8.$$

(b) Exhibit a polynomial $p(x) \in \mathbb{Z}[x]$ of degree 3 and roots α_1^2, α_2^2, α_3^2.

Solution : The elementary symmetric functions associated to the roots α_i^2 are

$$\sigma_1 = \sum \alpha_i^2 = 4 + \frac{5}{2}s_1 = 4,$$

$$\sigma_2 = \alpha_1^2\alpha_2^2 + \alpha_1^2\alpha_3^2 + \alpha_2^2\alpha_3^2 = \frac{1}{2}\left(\sigma_1^2 - \sum \alpha_i^4\right) = 4$$

$$\sigma_3 = s_3^2 = 25.$$

Set $p(x) = x^3 - 4x^2 + 4x - 25$.

3. Let $w_k = u_1^k + \cdots + u_n^k$.

(a) Prove Newton identities :

$$w_k - s_1 w_{k-1} + \cdots \pm s_{k-1}w_1 \mp k s_k = 0.$$

Solution : Consider the polynomial

$$f(x) = \prod_{i=1}^{n}(x - u_i) = \sum_{j=0}^{n}(-1)^j s_j x^{n-j}. \tag{1}$$

Then

$$f'(x) = \sum_{i=1}^{n} \frac{f(x)}{x - u_i} = f(x)\sum_{i=1}^{n}\left(\sum_{k\geqslant 0}\frac{u_i^k}{x^{k+1}}\right) = f(x)\sum_{k\geqslant 0}\frac{1}{x^{k+1}}\sum_{i=1}^{n}u_i^k =$$

$$= \left(\sum_{j=0}^{n}(-1)^j s_j x^{n-j}\right)\left(\sum_{k\geqslant 0}\frac{w_k}{x^{k+1}}\right) = \left(\sum_{j\geqslant 0}(-1)^j s_j x^{n-j}\right)\left(\sum_{k\geqslant 0}\frac{w_k}{x^{k+1}}\right)$$

[whereby $s_j = 0$ whenever $j > n$]

$$= \sum_{j+k=l}(-1)^j s_j w_k x^{n-(j+k)-1} = \sum_{l\geqslant 0}\left(\sum_{j\geqslant 0}(-1)^j s_j w_{l-j}\right)x^{n-l-1}$$

[whereby $w_{l-j} = 0$ whenever $j > l$]

$$= \sum_{l\geqslant 0}(-1)^l s_l (n-l)x^{n-l-1}$$

as derived from the right-hand side of (1). Hence, for each $l \geqslant 0$,

$$\sum_{j=0}^{l}(-1)^j s_j w_{l-j} = (-1)^l s_l (n-l).$$

If $l \geqslant n$, the right hand side is zero and

$$\sum_{j=0}^{l}(-1)^j s_j w_{l-j} = 0.$$

If $l < n$, then

$$\sum_{j=0}^{l}(-1)^j s_j w_{l-j} - (-1)^l s_l (n-l) = 0.$$

This proves all Newton identities.

(b) Do $w_1, \dots, w_n$ generate the ring of symmetric functions ?

Solution : From the Newton identities established in part (a), we observe that the symmetric polynomials

$$s_n = \frac{1}{n}\left(\sum_{j=0}^{n-1}(-1)^{n+j} s_j w_{n-j}\right)$$

may be defined recursively via solely the Newton sums w_k. Because of the presence of the integer denominator in the recursion formula, this definition is valid in the ring of symmetric functions.

4. Let $f(x) \in \mathbb{R}[x]$ be a monic polynomial of degree n, with roots $\alpha_1, \dots, \alpha_n$.

(a) Let N be the number of real roots of f. Show that

$$\begin{cases} N \equiv n \bmod 4 & \text{if } D(f) > 0 \\ N \equiv n-2 \bmod 4 & \text{if } D(f) < 0. \end{cases}$$

Solution : We identify the determining factors of the discriminant

$$D(f) = \prod_{i<j}(\alpha_i - \alpha_j)^2$$

as those of the form $\alpha_i - \overline{\alpha_i}$, for $\alpha_i \in \mathbb{C} \setminus \mathbb{R}$. If there are no such factors, then all roots are real and $N = n$. For every $\alpha \in \mathbb{C} \setminus \mathbb{R}$, then

$$\begin{aligned} D(f) &= (\alpha - \overline{\alpha})^2 \prod(\alpha - \alpha_i)^2 \prod(\overline{\alpha} - \alpha_i)^2 \prod(\alpha_i - \alpha_j)^2 \\ &= (\alpha - \overline{\alpha})^2 \prod |\alpha - \alpha_i|^4 \prod(\alpha_i - \alpha_j)^2 \end{aligned}$$

where the the first factor is negative.

Let T denotes the number of roots in $\mathbb{C} \setminus \mathbb{R}$ among α_i's. That is, $T + N = n$. This is always an even number, since for every complex root, its complex conjugate is also root of f. Hence there are $T/2$ factors of the form $\alpha - \overline{\alpha}$ in $D(f)$ and each contributes a minus sign. In particular, if $D < 0$, then $T/2$ must be odd, and $T \equiv 2 \bmod 4$. If $D > 0$, then $T/2$ is even and $T \equiv 0 \bmod 4$.

(b) How many real roots can $x^3 + px + q$ have ?

Solution : The discriminant of this polynomial is $-4p^3 - 27q^2$. Depending on the values of p and q, the polynomial has either 1 or 3 real roots by part (a).