D-MATH | Algebra II | FS 2014

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Serie 2

Fraction fields and maximal ideals

1. (a) Is there an integral domain that contains exactly 15 elements ?

   Solution : The only additive abelian group of order 15 is the cyclic group $\mathbb{Z}_{15}$ (you can see this with Sylow, and conclude that all groups of order 15 are isomorphic to the product $\mathbb{Z}_3 \times \mathbb{Z}_5$ of cyclic groups – cf. your notes of last semester or Section 7.7 in Artin). We show that in this ring, multiplication is the usual multiplication mod 15. However, $3 \cdot 5 \equiv 0 \mod 15$. This is a particular instance of the more general fact that the ring $\mathbb{Z}/(n)$ is an integral domain if and only if n is prime.

   (b) Let $\mathfrak{I} \subset R$ be an ideal. Prove that the quotient $R/\mathfrak{I}$ is an integral domain if and only if $\mathfrak{I}$ is a **prime ideal**. (An ideal $\mathfrak{I} \subsetneq R$ is prime if, for any two elements $a, b \in R$, $ab \in \mathfrak{I}$ then either $a \in \mathfrak{I}$ or $b \in \mathfrak{I}$.)

   Solution : Recall that an integral domain is, by definition, a non-zero ring, and $R/\mathfrak{I} \neq 0$ and $\mathfrak{I} \neq R$ are equivalent. Let $a, b \in R$. Since

   $$ab + \mathfrak{I} = (a + \mathfrak{I})(b + \mathfrak{I}),$$

   $ab \in \mathfrak{I}$ is equivalent to either $a + \mathfrak{I} = \mathfrak{I}$ or $b + \mathfrak{I} = \mathfrak{I}$, which are equivalent, respectively, to $a \in \mathfrak{I}$ and $b \in \mathfrak{I}$.

2. (a) Show that the quotient $R = \mathbb{Q}[x, y]/(x^2 + y^2 - 1)$ is an integral domain.

   Solution : We want to show that $(x^2 + y^2 - 1)$ is a prime ideal (cf. Exercise 1(b)). This is immediate since $x^2 + y^2 - 1$ can not be factorized in the product of two linear polynomials. We can see this geometrically : there is no way to make up a circle from two lines !

   (b) Write down a stereographic projection of the circle $x^2 + y^2 = 1$. Use it to show that the fraction field of R is isomorphic to the field of rational functions $\mathbb{Q}(t)$.

   Solution : We express the stereographic projection of the circle from the point $(0, 1)$ onto the affine line tangent at $(0, -1)$ in Cartesian coordinates :

   $$(x, y) \mapsto \frac{x}{1 - y}.$$

This map is bijective, with inverse

$$t \mapsto \left(\frac{2t}{t^2+1}, \frac{t^2-1}{t^2+1} \right).$$

Note that rational points are sent to rational points. In the language of algebraic geometry, we then say that the circle is birational to the affine line and that the circle is a rational curve.

Now, consider via the Substitution Principle (11.3.4 in Artin) the homomorphism from $\mathbb{Q}(t)$ to the fraction field of R that maps (the equivalence class of) $f(t)$ to (the equivalence class of) $f\left(\frac{x}{1-y}\right)$. It has inverse homomorphism sending (the equivalence class of) $f(x,y)$ to (the equivalence class of) $f\left(\frac{2t}{t^2+1}, \frac{t^2-1}{t^2+1}\right)$.

3. (a) Let $F[x]$ be a polynomial ring over a field F. Prove that the maximal ideals of $F[x]$ are the principal ideals generated by monic irreducible polynomials.

Proof : We already know that ideals in a polynomial ring over a field are principal ideals, and that any non-zero ideal is generated by the unique monic polynomial of lowest degree it contains (11.3.22 in Artin).

Let $(f(x))$ be a non-zero principal ideal. If $f(x)$ is factorizable, i.e. there exists non-constant polynomials $g(x), h(x) \in F[x]$ such that $f(x) = g(x)h(x)$, then $(f(x))$ is not maximal, since it is contained, e.g., in $(g(x))$. Conversely, if $(f(x))$ is contained in a larger ideal, then it can not be irreducible.

(b) Which principal ideals in $\mathbb{Z}[x]$ are maximal ?

Solution : A factorizable polynomial in $\mathbb{Z}[x]$ can not generate a maximal ideal, so let us consider the principal ideal generated by an irreducible polynomial $f(x)$. We show that, for any prime p that does not divide the leading coefficient of $f(x)$,

$$(f(x)) \subsetneq (p, f(x)) \subsetneq \mathbb{Z}[x].$$

The first inclusion is trivial. In the first exercise sheet, we have seen that the quotient rings $\mathbb{Z}[x]/(p, f(x))$ and $\mathbb{Z}_p[x]/(f(x))$ are isomorphic. This quotient can not be the zero ring since $(f(x))$ is maximal in $\mathbb{Z}_p[x]$. Hence, principal ideals in $\mathbb{Z}[x]$ can not be maximal.

(c) Find the maximal ideals of the following rings :

$$R_1 = \mathbb{R} \times \mathbb{R}, \quad R_2 = \mathbb{R}[x]/(x^2-3x+2), \quad R_3 = \mathbb{R}[x]/(x^2+x+1).$$

Solution : To first determine the ideals of $R_1 = \mathbb{R}\times\mathbb{R}$, consider the projection

$\mathbb{R} \times \mathbb{R} \to \mathbb{R}, (x, y) \mapsto x$. This is a surjective ring homomorphism with kernel $K = \{0\} \times \mathbb{R}$. Since $\mathbb{R}$ is a field, the kernel K is a maximal ideal in R_1 (by the Isomorphism Theorem). The projection on the second factor gives us the maximal ideal $\mathbb{R} \times \{0\}$ by a parallel argument. There can be no further ideal.

Next we apply the Correspondence Theorem to the quotient homomorphism

$$\mathbb{R}[x] \to \mathbb{R}[x]/(x^2 - 3x + 2) = R_2.$$

Since there is a factorisation

$$x^2 - 3x + 2 = (x - 2)(x - 1),$$

the ideals $(x - 2)$ and $(x - 1)$ are the only ones containing $(x^2 - 3x + 2)$, and these two ideals are clearly maximal, since $\mathbb{R}[x]/(x - 2) = \mathbb{R}$ is a field and similarly for $(x - 1)$. We now conclude by showing that maximal ideals are sent to maximal ideals under the quotient map. In fact, Let $\mathfrak{M}$ be a maximal ideal of R and $\mathfrak{I}$ an ideal such that $\mathfrak{I} \subset \mathfrak{M} \subset R$. It first follows from the mapping property of quotient rings, that there exists a surjective homomorphism $R/\mathfrak{I} \to R/\mathfrak{M}$. Its kernel is $\mathfrak{M}/\mathfrak{I}$. Then, by the first Isomorphism Theorem, we conclude that $R/\mathfrak{M}$ and $\frac{R/\mathfrak{I}}{\mathfrak{M}/\mathfrak{I}}$ are isomorphic. We can conclude that $\mathfrak{M}/\mathfrak{I}$ is a maximal ideal in $R/\mathfrak{I}$

In the last case, observe that $x^2 + x + 1$ is irreducible over $\mathbb{R}$, hence generates a maximal ideal (cf. (a)). Since R_3 is thus a field, its only ideals are $\{0\}$ and R_3.