D-MATH Algebra II FS 2014
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Solutions 6

1. Let α be a complex root of $x^3 - 3x + 4$. Find the inverse of $\alpha^2 + \alpha + 1$ in the form $a\alpha^2 + b\alpha + c$, with $a, b, c \in \mathbb{Q}$.

   Solution : Compute directly $(\alpha^2 + \alpha + 1)(a\alpha^2 + b\alpha + c) = 1$ using the relation $\alpha^3 = 3\alpha - 4$. You should get :

   $$(4a + b + c)\alpha^2 + (4b + c - a)\alpha + (c - 4a - 4b) = 1.$$

   Solving the resulting linear system

   $$\begin{aligned} 4a + b + c &= 0 \\ 4b + c - a &= 0 \\ c - 4a - 4b &= 1 \end{aligned}$$

   yields $3b = 2c - 1 = 5a$ and one should find $a = -\frac{1}{49}$.

2. (a) Show that $\sqrt{3} \notin \mathbb{Q}$, and $\sqrt{2} \notin \mathbb{Q}(\sqrt{3})$.

   Solution : Assume that $\sqrt{3} \in \mathbb{Q}$, then the polynomial $x^2 - 3$ is well defined over $\mathbb{Q}$. Applying Eisenstein's criterion, this polynomial is irreducible. This is a contradiction to $\sqrt{3} \in \mathbb{Q}$.
   Similarly, assume $\sqrt{2} \in \mathbb{Q}(\sqrt{3})$. Then $\sqrt{2}$ must have the form $a + b\sqrt{3}$, for some $a, b \in \mathbb{Q}$. It must follow that

   $$(a + b\sqrt{3})^2 = 2 \quad \text{and thus} \quad a^2 + 3b^2 - 2 + 2\sqrt{3}ab = 0.$$

   Since $(1, \sqrt{3})$ is a linear independent set (it is a basis for $\mathbb{Q}(\sqrt{3})$ as a vector field over $\mathbb{Q}$), either $a = 0$ or $b = 0$. Or, equivalently, either b is a root of $3x^2 - 2$ or a is a root of $x^2 - 2$. However, both polynomials are irreducible (you can see this with Eisenstein for instance) over $\mathbb{Q}$. This is a contradiction to $a, b \in \mathbb{Q}$.

   (b) Show that $\mathbb{Q}(\sqrt{2}, \sqrt{3}) = \mathbb{Q}(\sqrt{2} + \sqrt{3})$.

   Solution : One inclusion is trivial : $\mathbb{Q}(\sqrt{2}+\sqrt{3}) \subseteq \mathbb{Q}(\sqrt{2}, \sqrt{3})$, as $\sqrt{2}+\sqrt{3}$ is contained in $\mathbb{Q}(\sqrt{2}, \sqrt{3})$. To show the converse inclusion, we need to establish that $\sqrt{2}, \sqrt{3} \in \mathbb{Q}(\sqrt{2} + \sqrt{3})$. We have

   $$(\sqrt{2} + \sqrt{3})^2 = 5 + 2\sqrt{6}$$

hence $\sqrt{6} \in \mathbb{Q}(\sqrt{2}+\sqrt{3})$. One may then write

$$\sqrt{2} = \sqrt{2}+\sqrt{3} - \frac{\sqrt{6}}{\sqrt{2}}$$

which is equivalent to

$$\sqrt{2} = \frac{2+\sqrt{6}}{\sqrt{2}+\sqrt{3}} \in \mathbb{Q}(\sqrt{2}+\sqrt{3}).$$

A similar computation works to show $\sqrt{3} \in \mathbb{Q}(\sqrt{2}+\sqrt{3})$.

(c) Determine the degrees of the extensions $\mathbb{Q}(\sqrt{3})$ over $\mathbb{Q}$ and $\mathbb{Q}(\sqrt{2},\sqrt{3})$ over $\mathbb{Q}(\sqrt{3})$.

Solution : Since x^2-3 is the irreducible polynomial for $\sqrt{3}$ over $\mathbb{Q}$, we have $[\mathbb{Q}(\sqrt{3}) : \mathbb{Q}] = 2$. Similarly, x^2-2 is the irreducible polynomial for $\sqrt{2}$ over $\mathbb{Q}(\sqrt{3})$. Irreducibility in both cases follows from (a).

3. Let $\beta = \sqrt[3]{2}e^{2\pi i/3}$. Prove that $x_1^2 + \cdots + x_k^2 = -1$, $k \geqslant 1$, has no solutions with all $x_i \in \mathbb{Q}(\beta)$.

Solution : Let $\alpha = \sqrt[3]{2}$. Observe that both α and β are roots of $f(x) = x^3 - 2$. In fact, $f(x)$ is the irreducible polynomial for both α and β over $\mathbb{Q}$. We know that in this case $\mathbb{Q}(\alpha)$ and $\mathbb{Q}(\beta)$ are isomorphic (Proposition 15.2.8, Artin). Now, since $\mathbb{Q}(\alpha)$ is a subfield of $\mathbb{R}$, and the polynomial $x_1^2 + \cdots + x_k^2 + 1$ has no solution in $\mathbb{R}$, for whatever $k \geqslant 1$, the claim follows.

4. Let $K = F(\alpha)$ be a field extension generated by a transcendental element α, and let β be an element of K that is not in F. Prove that α is algebraic over the field $F(\beta)$.

Solution : We know that if α is transcendental over F, then $F(\alpha)$ is isomorphic to the field $F(x)$ of rational functions. Let $\beta = \frac{f(\alpha)}{g(\alpha)}$ be an element in $K = F(\alpha)$ but not in F. Then α is a root of t

$$p(x) = f(x) - \beta g(x) \in K[x].$$

We need to check that p is not trivially zero. The function $g(x)$ can, by definition, not be identically zero, that is for $g(x) = b_n x^n + \cdots + b_m x^m + \ldots b_0$, there is a coefficient $b_m \neq 0$. Then

$$p(x) = \cdots + (a_m - \beta b_m)x^m + \cdots + (a_0 - \beta b_0) = 0$$

requires $\beta = a_m/b_m$ and this is not possible as we assume $\beta \notin F$.