Solutions 7

Degree of field extension, irreducible polynomial

1. (a) Let F be a field, and let α be an element that generates a field extension of F of degree 5. Prove that α^2 generates the same extension.

   Solution : The element α generates $F(\alpha)$ with $[F(\alpha) : F] = 5$. Since $\alpha^2 \in F(\alpha)$, $F(\alpha^2) \subset F(\alpha)$. Because $[F(\alpha) : F(\alpha^2)]$ must divide $[F(\alpha) : F] = 5$ and $\alpha \notin F$, we can conclude that $F(\alpha^2) = F(\alpha)$.

   (b) Prove the last statement for 5 replaced by any odd integer.

   Solution : If $[F(\alpha) : F(\alpha^2)]$ divides an odd integer, then it must be odd itself. The irreducible polynomial for α over $F(\alpha^2)$ must also divide the polynomial $x^2 - \alpha$. Hence the degree is an odd integer that is less or equal to 2 and we can conclude that $F(\alpha^2) = F(\alpha)$.

2. Prove that $x^4 + 3x + 3$ is irreducible over $\mathbb{Q}[\sqrt[3]{2}]$.

   Solution : Applying the Eisenstein criterium, we see that the polynomial

   $$f(x) = x^4 + 3x + 3$$

   is irreducible over Q. Let α be a root of $f(x)$. Then α defines a field extension of degree 4 over $\mathbb{Q}$. By coprimality of the degrees, the extension $\mathbb{Q}(\alpha, \sqrt[3]{2})$ over $\mathbb{Q}$ has degree 12. The irreducible polynomial for α over $\mathbb{Q}(\sqrt[3]{2})$ must have degree 4. Hence it is $f(x)$.

3. Let $K = \mathbb{Q}(\alpha)$ where α is a root of $x^3 - x - 1$. Determine the irreducible polynomial for $1 + \alpha^2$ over $\mathbb{Q}$.

   Solution : We compute

   $$(1 + \alpha^2)^2 = 3\alpha^2 + \alpha + 1, \qquad (1 + \alpha^2)^3 = 7\alpha^2 + 5\alpha + 2.$$

   Getting rid of the factors α then leads to

   $$(1 + \alpha^2)^3 - 5(1 + \alpha^2)^2 + 8(1 + \alpha^2) - 5 = 0.$$

   We conclude by showing that $f(x) = x^3 - 5x^2 + 8x - 5$ is irreducible over $\mathbb{Q}$. This follows e.g. because $x^3 + x^2 + 1$ is irreducible in $\mathbb{F}_2[x]$.

4. Determine the irreducible polynomials for $\alpha = \sqrt{3} + \sqrt{5}$ over the following fields

$$\mathbb{Q}, \quad \mathbb{Q}[\sqrt{5}], \quad \mathbb{Q}[\sqrt{10}], \quad \mathbb{Q}[\sqrt{15}].$$

Solution : We have

$$\alpha^2 = 8 + 2\sqrt{15}, \quad (\alpha^2 - 8)^2 = 60, \quad (\alpha - \sqrt{5})^2 = 3.$$

Therefore

- $x^4 - 16x^2 + 4$ is the irreducible polynomial for α over $\mathbb{Q}$,
- $x^2 - 2\sqrt{5}x + 2$ is the irreducible polynomial for α over $\mathbb{Q}[\sqrt{5}]$, since there can be no monic linear polynomial in $\mathbb{Q}[\sqrt{5}]$ with root α.
- $x^2 - 8 - 2\sqrt{15}$ is the irreducible polynomial for α over $\mathbb{Q}[\sqrt{15}]$ following the same argumentation as above,
- and since there are no linear or quadratic polynomial with root α over $\mathbb{Q}[\sqrt{10}]$, the irreducible polynomial is in this case also $x^4 - 16x^2 + 4$.

5. A field extension K/F is an algebraic extension if every element of K is algebraic over F.

   (a) Let L/K and K/F be algebraic extensions. Prove that L/F is an algebraic extension.

   Solution : Let l be an element of L. By assumption, L/K is algebraic, so l is the root of a non-zero polynomial over K. We show that it is also the root of a non-zero polynomial with coefficients in F.

   Let $f(x) = a_n x^n + \cdots + a_0$ be the polynomial in $K[x]$ with root l. Since we also assume that the extension K/F is algebraic , the degree $[F(a_n, \ldots, a_0) : F]$ is finite. If we now consider $F' = F(a_n, \ldots, a_0, l)$, the degree $[F' : F]$ is finite.

   (b) Let $\alpha, \beta \in \mathbb{C}$. Prove that if $\alpha + \beta$ and $\alpha\beta$ are algebraic numbers, then α and β are also algebraic numbers.

   Solution : If both $\alpha + \beta$ and $\alpha\beta$ are algebraic, then the extension $K = F(\alpha + \beta, \alpha\beta)$ is algebraic over F, following the definition above. We show that $F(\alpha, \beta)/K$ is algebraic and conclude with (a). For this, note that the polynomial

   $$x^2 - (\alpha + \beta)x + \alpha\beta$$

   has coefficients in K and has both α and β as roots.