Solutions 8

Constructions with Ruler and Compass

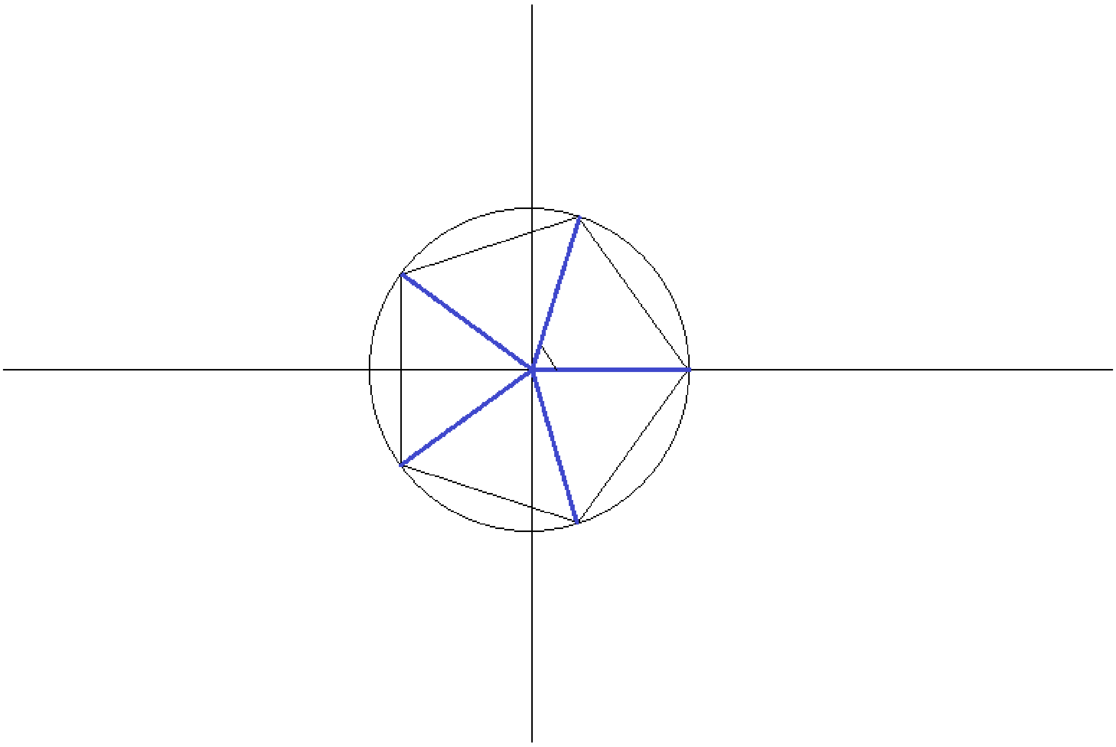
1. Express $\cos 15^o$ in terms of real square roots.

   Solution : We have $\alpha := \cos 15^o = \cos\frac{\pi}{12} = \frac{\sqrt{6}+\sqrt{2}}{4}$. Then $\alpha^2 = \frac{2+\sqrt{3}}{4} = \frac{1+\sqrt{3}/2}{2}$, and we can write
   $$\alpha = \sqrt{\frac{1+\sqrt{3}/2}{2}}.$$

2. Prove that the regular pentagon can be constructed by ruler and compass, by field theory, and by finding an explicit construction.

   Solution : Set $\alpha = \frac{2\pi}{5}$ and $z = e^{i\alpha} = \cos(\alpha) + i\sin(\alpha) =: x + iy$. Then developing
   $$z^5 = (x+iy)^5 = 1$$
   into
   $$x^5 + 5xy^4 - 10x^3y^2 = 1$$
   and using $y^2 = 1 - x^2$, one gets that $\cos\alpha$ is a root of the polynomial
   $$16x^5 - 20x^3 + 5x - 1.$$
   Clearly 1 is another root and we can further factorize
   $$\begin{aligned} 16x^5 - 20x^3 + 5x - 1 &= (x-1)(16x^4 + 16x^3 - 4x^2 - 4x + 1) \\ &= (x-1)(4x^2 + 2x - 1)^2 \end{aligned}$$
   so that, by the quadratic equation and the fact that $\alpha = \cos(2\pi/5) > 0$, we get
   $$\alpha = \frac{-1+\sqrt{5}}{4}.$$
   Hence α is constructible over $\mathbb{Q}$.

   Now to explicitly construct the regular pentagon, start from the points (0,0) and (0,1), and build successively $2\pi/5$ angles, every time constructing the points $p_n = (\cos(2\pi n/5), \sin(2\pi n/5))$,$n = 1, \ldots, 4$, and taking the half-lines through the origin and each p_n (in blue in the figure below).



Take the circle of radius 1 centred at the origin and construct the points at the intersection of the circle and the half-lines. Connecting these yield the regular pentagon.

3. Is the regular 9-gon constructible by ruler and compass ?

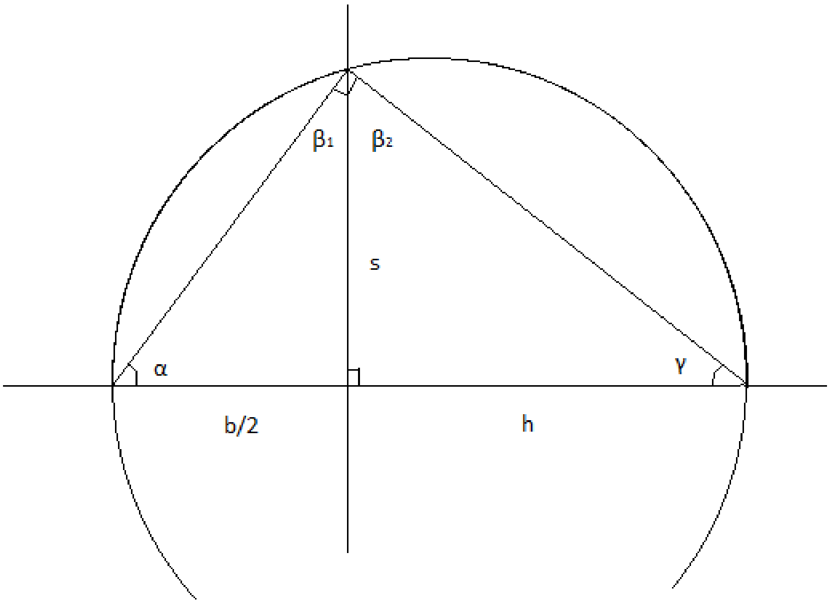
   Solution : You have seen in class that $\cos(\pi/9)$ is not constructible. Applying $cos^2(\alpha) = \frac{1+\cos(2\alpha)}{2}$, we get

   $$\cos(2\pi/9) = 2\left(\cos(\pi/9)\right)^2 - 1.$$

   It follows that $\cos(2\pi/9)$ is not constructible.

4. Is it possible to construct a square whose area is that of a given triangle ?

   Solution : Let Δ denote the given triangle. As area$(\Delta) = \frac{1}{2}bh$, where b refers to the base of Δ and h to its height, we construct a length s such that $s^2 = \frac{1}{2}bh$. We know how to mark off lengths, here $b/2$ and h, onto a constructed line so that the two segments are adjacent. Then take the circle passing through the endpoints of diameter $b/2 + h$. and then the triangle inscribed in this circle with edges the endpoints.



We can now check that the height s of this triangle is exactly the length we are looking for by checking

$$\frac{s}{h} = \frac{b/2}{s}.$$

Denote the angles α, β, γ, where $\beta = \pi/2$ breaks down in $\beta = \beta_1 + \beta_2$ for the triangle on the left and the triangle on the right. We then observe

$$\frac{s}{h} = \tan\gamma = \tan(\pi/2 - \beta_2) = \tan(\beta_1) = \frac{b/2}{s}.$$

5. Thinking of the plane as the complex plane, describe the set of constructible points as complex numbers.

   Solution : Thinking of the plane as the complex plane means that a complex number $z = x + iy$ is constructible if the coordinate point (x, y) is constructible in the plane. Recall that a real number x is constructible if the point $(0, x)$ is constructible. If z is constructible, then taking the line passing through (x, y) that is parallel to the vertical axis allows to construct $(x, 0)$, and by taking the parallel to the x-axis passing through (x, y) gives $(0, y)$. Conversely, if x and y are constructible reals, then taking the perpendicular at $(x, 0)$ to the x-axis, the perpendicular at $(0, y)$ to the y-axis, their intersection point is constructible, hence z is constructible. Hence $z = x + iy$ is constructible if and only if both x and y are constructible.