Exercise Sheet 2

Exercise 1

Let G be a locally compact, Hausdorff, Lindelöf group and let X be a Hausdorff Baire space. Further, let $\alpha : G \times X \to X$ be a continuous, transitive action of G on X. Given $x \in X$, show that the map $G/\operatorname{stab}_G(x) \to X,\ [g] \mapsto gx$ is a homeomorphism.

The Lindelöf property follows e.g. from second-countability, or σ-compactness. Prime examples of Baire spaces are complete metric spaces and locally compact Hausdorff spaces. In particular, a globally symmetric space M is homeomorphic to $\operatorname{Iso}(M)/K$, where K is a point stabilizer, by this exercise.

Exercise 2

Let G be a Lie group with Lie algebra $\mathfrak{g}$. Choose neighbourhoods $U_0 \in \mathcal{U}(0 \in \mathfrak{g})$ and $U \in \mathcal{U}(e \in G)$ such that $\exp|_{U_0} : U_0 \to U$ is a diffeomorphism. Further, let $V \subseteq U$ be open such that $V^2 \subseteq U$ and define $V_0 := \exp^{-1}(V)$. Consider the following version of the Baker-Campbell Hausdorff theorem:

The map $\eta : V_0 \times V_0 \to U_0,\ (X, Y) \mapsto \exp^{-1}(\exp(X)\exp(Y))$ is real-analytic.

Using this theorem, show that G admits a compatible, real-analytic structure such that multiplication and inversion are real analytic maps.

Exercise 3

Let M be a locally symmetric space with universal cover $\widetilde{M}$.

(i) Show that $M \cong \pi_1(M)\backslash \operatorname{Iso}(\widetilde{M})/K$ where K is a point stabilizer in $\operatorname{Iso}(\widetilde{M})$.

(ii) Exhibit the torus, the Klein bottle and real projective space in this fashion.

Part (i) shows that the study of locally symmetric spaces is equivalent to the study of certain Lie groups and certain subgroups of the latter.