Exercise Sheet 3

Exercise 1

Let X be a globally symmetric, simply connected space and let $p \in X$. Show that $K := \mathrm{stab}_{\mathrm{Iso}(X)^\circ}(p)$ is connected.

This may not be the case if X is not simply connected.

Exercise 2

This exercise provides a formula for the derivative of the Riemannian exponential map associated to an arbitrary connection, at points other than the origin of the tangent space in question.

Let M be an analytic manifold with an analytic connection ∇. Fix a point $p \in M$ and a normal neighbourhood N_p of $p \in M$. Given $X \in \mathrm{T}_pM - \{0\}$, define a vector field X^* on N_p by $X_q^* := P_{\gamma_{pq};0,1}$ $(q \in N_p)$ where $\gamma_{pq} : [0,1] \to M$ is the unique geodesic with respect to ∇ in N_p joining p and q, and $P_{\gamma_{p,q};0,1}$ denotes parallel transport with respect to ∇ along γ_{pq}. Consider the exponential map $\mathrm{Exp} := \mathrm{Exp}_p : N_0 \to N_p$ associated to ∇ where N_0 is the normal neighbourhood of $0 \in \mathrm{T}_pM$ associated to N_p. Going through the steps below, show that there is $\varepsilon > 0$ such that for $|t| < \varepsilon$ and $Y \in \mathrm{T}_{tX}(\mathrm{T}_pM) \cong \mathrm{T}_pM$:

$$D_{tX}\,\mathrm{Exp}(Y) = \left(\sum_{n=0}^{\infty} \frac{\theta(-tX^*)^n}{(n+1)!} Y^*\right)_{\mathrm{Exp}\,tX}$$

where, given a manifold N, we define $\theta : \mathrm{Vect}(N) \to \mathrm{L}(\mathrm{Vect}(N)),\ X \mapsto [X, -]$.

Note that the above formula reduces to $D_{tX}\,\mathrm{Exp}(Y) = Y$ for $t = 0$.

(i) Let $f : U \to \mathbb{R}$ be an analytic function on $U \in \mathcal{U}(p)$. Show that there is a neighbourhood $U_0 \in \mathrm{T}_pM$ such that for all $Z \in U_0$:

$$f(\mathrm{Exp}\,Z) = \sum_{n=0}^{\infty} \frac{((Z^*)^n f)(p)}{n!}.$$

(ii) Let f be as in part (i). Show that for $Y \in \mathrm{T}_pM$ and $t \in \mathbb{R}$ small we have

$$D_{tX}\,\mathrm{Exp}(Y)f = \sum_{n=0}^{\infty} \frac{t^n}{(n+1)!} \left(\left(\sum_{k=0}^{n} (X^*)^{n-k} Y^* (X^*)^k\right) f\right)(p).$$

(iii) Rearrange the expression obtained in part (ii) to

$$D_{tX}\,\mathrm{Exp}(Y)f = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \left(\frac{(tX^*)^{n-k}}{(n-k)!} \frac{(\theta(-tX^*)^k}{(k+1)!} Y^*\right) f\right)(p).$$

(iv) Argue that the expression obtained in part (iii) can be rewritten as

$$D_{tX}\,\mathrm{Exp}(Y)f = \sum_{n=0}^{\infty} \left(\frac{(tX^*)^n}{n!} \left(\sum_{k=0}^{\infty} \left(\frac{\theta(-tX^*)^k}{(k+1)!} Y^*\right) f\right)\right)(p)$$

for sufficiently small $t \in \mathbb{R}$. Conclude the overall assertion using part (i).