Exercise Sheet 4

Exercise 1

Let M be a globally symmetric space and $p \in M$. Denote by $s_p : M \to M$ the geodesic symmetry about $p \in M$. Define $G := \mathrm{Iso}(G)^\circ$ and $K := \mathrm{stab}_G(p)$, as well as $\sigma : G \to G,\ g \mapsto s_p g s_p$ and $\pi : G \to M,\ g \mapsto gp$. Then for $\mathfrak{p} := \mathrm{E}_{D_e\sigma}(-1)$ the following diagram commutes:

$$\begin{CD} \mathfrak{p} @>{D_e\pi|_{\mathfrak{p}}}>> T_pM \\ @V{\exp|_{\mathfrak{p}}}VV @VV{\mathrm{Exp}_p}V \\ G @>>{\pi}> M. \end{CD}$$

Make this explicit for $M \in \{\mathbb{R}^2, \mathbb{S}^2, \mathbb{H}^2\}$ and check commutativity for some non-trivial elements of the respective $\mathfrak{p}$.

Exercise 2

Review the lecture's proof of the following theorem and fill in details where necessary:
Let M be a globally symmetric space and $p \in M$. Denote by $s_p : M \to M$ the geodesic symmetry about $p \in M$. Define $G := \mathrm{Iso}(G)^\circ$ and $K := \mathrm{stab}_G(p)$, as well as $\sigma : G \to G,\ g \mapsto s_p g s_p$ and $\pi : G \to M,\ g \mapsto gp$. Let $\mathfrak{p} := \mathrm{E}_{D_e\sigma}(-1)$.

(i) If $\mathfrak{n} \subseteq \mathfrak{p}$ is a Lie triple system then $N := \mathrm{Exp}_p(D_e\pi(\mathfrak{p}))$ is a totally geodesic submanifold of M.

(ii) If N is a totally geodesic submanifold through p then $(D_e\pi|_{\mathfrak{p}})^{-1}(T_pN) \subseteq \mathfrak{p}$ is a Lie triple system.

Exercise 3

Let M be a globally symmetric space and let $N \subseteq M$ be a totally geodesic submanifold. Show that N is a globally symmetric space in its own right. Determine the Riemannian symmetric pair associated to N.