Exercise Sheet 5

Exercise 1

Let $G \leq \mathrm{SL}(n, \mathbb{R})$ be a closed connected subgroup which is stable under transposition. Then $\sigma : G \to G,\ g \mapsto {g^{-1}}^T$ is in involutive automorphism of G. Set $K := G^\sigma = G \cap \mathrm{SO}(n)$. Then (G, K, σ) is a Riemannian symmetric pair. Show that G/K is a totally geodesic submanifold of

$$\mathrm{Sym}_1^+(n) := \{X \in M_{n,n}(\mathbb{R}) \mid X^T = X,\ X \gg 0,\ \det X = 1\}.$$

Exercise 2

Let G be a Lie group with Lie algebra $\mathfrak{g}$. Suppose $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ is the direct sum of two ideals $\mathfrak{g}_1$ and $\mathfrak{g}_2$. Further, let $\mathfrak{k}_1$ and $\mathfrak{k}_2$ be subalgebras of $\mathfrak{g}_1$ and $\mathfrak{g}_2$ respectively. Set $\mathfrak{k} := \mathfrak{k}_1 + \mathfrak{k}_2$. Show that $\mathfrak{k}$ is compactly embedded in $\mathfrak{g}$ if and only if $\mathfrak{k}_1$ and $\mathfrak{k}_2$ are compactly embedded in $\mathfrak{g}_1$ and $\mathfrak{g}_2$ respectively.

Apply this in the proof of the decomposition theorem for orthogonal symmetric Lie algebras.

Exercise 3

Review the lecture's proof of the following theorem and fill in details where necessary:
Let M be a simply connected symmetric space. Then $M = M_0 \times M_+ \times M_-$ is a Riemannian product of symmetric spaces M_0, M_+ and M_- of Euclidean, non-compact and compact type respectively.