Exercise Sheet 6

Exercise 1

Show that the pair $(\mathfrak{sl}(n,\mathbb{R}),\theta)$, where $\theta:\mathfrak{sl}(n,\mathbb{R})\to\mathfrak{sl}(n,\mathbb{R}),\ X\mapsto -X^T$, is an *irreducible* orthogonal symmetric Lie algebra

Exercise 2

Let $(\mathfrak{g},\theta)$ be an orthogonal symmetric Lie algebra with Cartan decomposition $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}$. Show that its dual $(\mathfrak{g},\theta)^*:=(\mathfrak{g}^*,\zeta)$ where $\mathfrak{g}^*:=\mathfrak{k}+i\,\mathfrak{p}\leq_{\mathbb{R}}\mathfrak{g}_{\mathbb{C}}$ and $\zeta:\mathfrak{g}^*\to\mathfrak{g}^*,\ X+iY\mapsto X-iY$ is an orthogonal symmetric Lie algebra as well.

Exercise 3

Consider the orthogonal symmetric Lie algebra $(\mathfrak{so}(n),\theta_{p,q}|_{\mathfrak{so}(n)})$ where

$$\theta_{p,q}:\mathfrak{gl}(n,\mathbb{R})\to\mathfrak{gl}(n,\mathbb{R}),\ X\mapsto I_{p,q}XI_{p,q},\quad \text{with}\quad I_{p,q}:=\begin{pmatrix}-I_p & \\ & I_q\end{pmatrix},\ p+q=n.$$

Show that $(\mathfrak{so}(n),\theta_{p,q}|_{\mathfrak{so}(n)})^*\cong(\mathfrak{so}(p,q),\theta|_{\mathfrak{so}(p,q)})$.

In particular, $\mathbb{S}^n$ *and* $\mathbb{H}^n$ *are dual symmetric spaces.*