R. Hiptmair
S. Hosseini
C. Pagliantini

Homework Problem Sheet 2

Introduction. Solution of linear systems of equations and condition number.

Problem 2.1 Solving A System of Linear Equations with Rounding

In [NMI, Sect. 2.5] it was demonstrated that roundoff can cause instability of Gaussian elimination, unless a suitable pivot policy is implemented. This problem examines this effect in detail for a small example, similar to [NMI, Ex. 2.13] and [NMI, Ex. 2.25]. You are advised to study these examples again before tackling this problem.

Let $\mathbf{A} \in \mathbb{R}^{2\times 2}$ and $\mathbf{b} \in \mathbb{R}^2$ be given by

$$\mathbf{A} = \begin{pmatrix} 0.005 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 0.5 \\ 1 \end{pmatrix}.$$

We will solve the system $\mathbf{A}\mathbf{x} = \mathbf{b}$ for $\mathbf{x}$ using Gaussian elimination in different ways:

(2.1a) Without rounding errors.

(2.1b) Without pivoting, i.e. without interchanging rows or columns, in the floating-point arithmetic $\mathbb{F}(10, 3, -10, 10)$ up to three significant digits.

(2.1c) With pivoting in the floating-point arithmetic $\mathbb{F}(10, 3, -10, 10)$.

(2.1d) Compare and comment on the above results.

Remark: Calculations in floating-point arithmetic $\mathbb{F}$ are meant as follows: the results of elementary operations from $\{+, -, \cdot, /\}$ are calculated exactly but rounded to a number in $\mathbb{F}$ before being used for further calculations, see [NMI, Ch. 1].

Problem 2.2 LU-Decomposition of Symmetric Matrices

In [NMI, Sect. 2.2] you learned about the LU-decomposition of square matrices as a numerically equivalent way to express Gaussian elimination. For symmetric matrices certain simplifications of the algorithms are possible and these will be explored in this problem.

Let the matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$ be symmetric and assume there exist matrices $\mathbf{L}, \mathbf{U} \in \mathbb{R}^{n\times n}$ such that $\mathbf{A} = \mathbf{L}\mathbf{U}$ and that $\mathbf{L}$ and $\mathbf{U}$ result from the $\mathbf{L}\mathbf{U}$ decomposition of $\mathbf{A}$ *without pivoting*.

(2.2a) Let $\mathbf{L}_1^{-1}$ be the matrix of the first step of the $\mathbf{L}\mathbf{U}$ decomposition with entries $l_{i,1} := a_{i,1}/a_{11}$, $i = 2, \dots, n$, ones on the diagonal and zeros elsewhere, such that $\mathbf{L}_1^{-1}\mathbf{A}^{(1)} = \mathbf{A}^{(0)} = \mathbf{A}$, where

$$\mathbf{A}^{(1)} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & & & \\ \vdots & & \mathbf{B}^{(1)} & \\ 0 & & & \end{pmatrix}.$$

Show that the Matrix $\mathbf{B}^{(1)} \in \mathbb{R}^{(n-1)\times(n-1)}$ is symmetric again.

(2.2b) Below, you can see the algorithm for the *in situ* $\mathbf{LU}$ decomposition without pivoting. In situ means that the system matrix $\mathbf{A}$ is replaced with its LU-factors by storing $\mathbf{U}$ in the upper triangular part of $\mathbf{A}$, whereas its (strict) lower triangular part is replaced with $\mathbf{L}$. Note that the diagonal of $\mathbf{L}$ need not be stored, because it is all 1!

Show that the algorithm below incurs asymptotic computational cost of $\frac{2}{3}n^3 + \mathcal{O}(n^2)$ for $n \to \infty$. Assume that each arithmetic operation from $\{+, -, \cdot, /\}$ on matrix entries induces unit costs. Ignore costs for loop indices.

for $k = 1, \dots, n-1$ **do**
    for $i = k+1, \dots, n$ **do**
        $a_{ik} \leftarrow \dfrac{a_{ik}}{a_{kk}}$
        for $j = k+1, \dots, n$ **do**
            $a_{ij} \leftarrow a_{ij} - a_{ik}a_{kj}$
        end for
    end for
end for

(2.2c) Change the algorithm from subproblem (2.2b) to an in situ $\mathbf{LU}$ decomposition without pivoting for symmetric matrices making use of the symmetry such that the resulting runtime complexity is $\frac{1}{3}n^3 + \mathcal{O}(n^2)$. Prove that your algorithm indeed has this complexity. Thereto, assume the same unit costs as above.

Problem 2.3 Condition Number

In [NMI, Sect. 2.4] you learned about the concept of various *condition numbers* of a mapping $F : X \to Y$, where X and Y are normed vector spaces. You also saw how condition numbers can be computed for continuously differentiable F based on the norm of its derivative. These formulas will have to be applied in this problem.

The relative condition number of a function measures how sensitive the function is to errors in the input and how a small error in the input influences the error in the output. According to definition [NMI, Def. 2.19], the condition of a function $f : \mathbb{R} \to \mathbb{R}$ at a point x is given by

$$\mathrm{cond}(f, x) := \frac{|f'(x)|}{|f(x)|}|x|.$$

For a real number $p \geqslant 2$, calculate the condition numbers $\kappa_{1,2} = \mathrm{cond}(y_{1,2}, p)$ of the roots $y_{1,2}(p)$ of the quadratic polynomial $y^2 - py + 1$ and express them as a function of $y_{1,2}$. Note that the solutions of the quadratic equation $y^2 - py + 1 = 0$ are given by

$$y_{1,2}(p) = \frac{p \pm \sqrt{p^2 - 4}}{2}. \tag{2.3.1}$$

HINT: You may want to use Viète's formulas, i. e. $y_1 + y_2 = p$ and $y_1 y_2 = 1$.

Problem 2.4 An Estimate of the Condition Number

In [NMI, Sect. 2.4.3] we conducted a careful analysis of the sensitivity of the solution map for linear systems of equations with respect to perturbations of the right hand side vector and the system matrix. Neumann series arguments and norm estimates for inverses were important tools. This problem will revisit them.

Let $\mathbf{A}, \Delta\mathbf{A} \in \mathbb{C}^{n\times n}$ be matrices such that $\mathbf{A}$ is regular. Prove the following properties:

(2.4a) If $\|\Delta\mathbf{A}\|_2 \|\mathbf{A}^{-1}\|_2 < 1$, then $\mathbf{A} + \Delta\mathbf{A}$ is regular.

(2.4b) If $\mathbf{B} \in \mathbb{C}^{n\times n}$ is singular, then $1 \leqslant \|\mathbf{A}^{-1}\|_2 \|\mathbf{A} - \mathbf{B}\|_2$.

(2.4c) For the condition number in the 2-norm, namely $\kappa_2(\mathbf{A}) := \|\mathbf{A}\|_2 \|\mathbf{A}^{-1}\|_2$, we have

$$\kappa_2(\mathbf{A})^{-1} \leqslant \inf\left\{ \frac{\|\mathbf{A} - \mathbf{B}\|_2}{\|\mathbf{A}\|_2} \,\middle|\, \mathbf{B} \in \mathbb{C}^{n\times n} \text{ singular} \right\}.$$

Published on March 4, 2014.

To be submitted on March 11/12, 2014.

MATLAB: Submit all files in the online system. Include the files that generate the plots. Label all your plots. Include commands to run your functions. Comment on your results.

Written exercises: Hand-in the solutions during the exercise class or in the labeled boxes in HG G 53.x.

References

[NMI] Lecture Slides for the course "Numerical Analysis I".

Last modified on March 4, 2014