R. Hiptmair
S. Hosseini
C. Pagliantini

Homework Problem Sheet 4

Introduction. This problem sheet is devoted to matrix norms and matrices with special structures.

Problem 4.1 Matrix Norms

In [NMI, Sect. 0.10] the singular value decomposition (SVD) of a matrix $\mathbf{A} \in \mathbb{C}^{m\times n}$ is presented. The SVD is a very versatile tool in numerical analysis. For instance, it can be used to express matrix norms through the singular values, which is the focus of this problem.

For the remainder of this problem, let $\mathbf{A}$ be a square matrix, $\mathbf{A} \in \mathbb{C}^{n\times n}$.

(4.1a) Review the definition of the SVD as given in [NMI, Thm. 0.62].

(4.1b) Show that $\|\mathbf{A}\|_2 = \max_{i=1,\dots,n} \sigma_i$, where σ_i is the i^{th} singular value of $\mathbf{A}$.

(4.1c) Prove that the Frobenius norm of $\mathbf{A}$ satisfies $\|\mathbf{A}\|_F = \left(\sum_{i=1}^n \sigma_i^2\right)^{1/2}$.

(4.1d) Let $\mathbf{A} \in \mathbb{C}^{n\times n}$ be a symmetric positive definite matrix. Prove that

$$\|\mathbf{A}\|_2 = \sup_{\mathbf{x}\neq 0} \frac{\mathbf{x}^{\mathsf{H}}\mathbf{A}\mathbf{x}}{\mathbf{x}^{\mathsf{H}}\mathbf{x}}.$$

HINT: The “singular value decomposition” of a symmetric matrix has a special form.

(4.1e) Let $\mathbf{A} = \mathbf{L}\mathbf{L}^{\mathsf{H}}$ be the Cholesky decomposition of the symmetric positive definite matrix $\mathbf{A} \in \mathbb{C}^{n\times n}$, as presented in [NMI, Sect. 2.7]. Prove that $\|\mathbf{A}\|_2 = \|\mathbf{L}\|_2^2$.

Problem 4.2 LU-decomposition of Strictly Diagonally Dominant Matrices

Symmetric positive definite matrices allow stable Gaussian elimination without pivoting. There are other important classes of matrices that enjoy this property and in this problem you are going to see one.

Definition. A matrix $\mathbf{A} = (a_{ij})_{i,j=1}^n \in \mathbb{C}^{n\times n}$ is *strictly (column) diagonally dominant* if for every column of the matrix the magnitude of the diagonal entry is larger than the sum of the magnitudes of all the other (non-diagonal) entries in that column, that is

$$|a_{jj}| > \sum_{j\neq i=1}^{n} |a_{ij}| \quad \text{for all } j = 1,\dots,n\ . \tag{4.2.1}$$

(4.2a) Show that every symmetric strictly diagonally dominant matrix is regular.

HINT: Consider a kernel vector of $\mathbf{A}$ and its component with the largest magnitude.

(4.2b) Show that the Gaussian elimination with partial pivoting (Spaltenpivotsuche) as introduced in [NMI, Sect. 2.5] of the lecture notes, of a strictly diagonally dominant matrix always selects the element a_{11} as pivot element at the first step.

(4.2c) Let $\mathbf{A}^{(1)}$ denote the matrix that emerges from the first step of Gaussian elimination of $\mathbf{A} \in \mathbb{C}^{n\times n}$, that is, after those row transformations that make the subdiagonal entries of the first column vanish.

Show that $\mathbf{A}^{(1)}(2{:}n, 2{:}n)$ (MATLAB notation for the selection of a sub-matrix) is strictly diagonally dominant, if $\mathbf{A}$ enjoys this property.

HINT: Start with a formula for the entries of $\mathbf{A}^{(1)}$.

(4.2d) Show that the Gaussian elimination with partial pivoting of a strictly diagonally dominant matrix does not trigger any row swaps.

HINT: Use the result of the two previous subproblems and perform induction with respect to the matrix size.

Problem 4.3 Schur Complement

The so-called Schur complement plays a central role in many algorithms of numerical linear algebra. It is defined as follows. Suppose $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$ are respectively $p \times p$-, $p \times q$-, $q \times p$- and $q \times q$-matrices, and that $\mathbf{A}$ is invertible. Then the Schur complement of the block $\mathbf{A}$ of the matrix

$$\mathbf{M} := \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix}$$

is the $q\times q$-matrix $\mathbf{S} = \mathbf{D} - \mathbf{C}\mathbf{A}^{-1}\mathbf{B}$. In this problem assume that $\mathbf{M} \in \mathbb{R}^{(p+q)\times(p+q)}$ is symmetric positive definite.

(4.3a) Let $\mathbf{S} \in \mathbb{R}^{q\times q}$ be symmetric and positive definite, and $\mathbf{b} \in \mathbb{R}^q$. Show that the vector $\mathbf{x}^* := \mathbf{S}^{-1}\mathbf{b}$ is the unique minimizer of the function

$$f : \left\{ \begin{array}{ccc} \mathbb{R}^p & \rightarrow & \mathbb{R} \\ \mathbf{x} & \rightarrow & \frac{1}{2}\mathbf{x}^\top \mathbf{S}\mathbf{x} - \mathbf{b}^\top \mathbf{x} \end{array} \right. . \tag{4.3.1}$$

HINT: Find an equivalent expression for $f(\mathbf{x}) - f(\mathbf{x}^*)$ that is guaranteed to be positive for $\mathbf{x} \neq \mathbf{x}^*$. To that end remember what it means that $\mathbf{S}$ is positive definite (SPD).

(4.3b) Prove that

$$\mathbf{y}^T\mathbf{S}\mathbf{y} = \min_{\mathbf{x}\in\mathbb{R}^p} \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix}^T \mathbf{M} \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix}, \quad \mathbf{y} \in \mathbb{R}^q .$$

HINT: The expression, of which we take the infimum, is structurally close to f from (4.3.1). Hence, the result of (4.3a) can be used.

(4.3c) Prove that $\mathbf{S}$ is symmetric positive definite.

(4.3d) Prove that

$$\kappa_2(\mathbf{S}) \leq \kappa_2(\mathbf{M}).$$

Problem 4.4 Solving Band Systems

This exercise aims at showing that considerable gain in efficiency can often be achieved by exploiting the special structure of a linear system of equations with a system matrix that features a special *band structure*.

The banded matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$, given by the three vectors $\mathbf{u} \in \mathbb{R}^{n-2}$, $\mathbf{v} \in \mathbb{R}^{n}$ and $\mathbf{w} \in \mathbb{R}^{n-1}$, has the following form:

$$\mathbf{A} = \begin{pmatrix} v_1 & w_1 & & & \mathbf{0} \\ 0 & \ddots & \ddots & & \\ u_1 & \ddots & \ddots & \ddots & \\ & \ddots & \ddots & \ddots & w_{n-1} \\ \mathbf{0} & & u_{n-2} & 0 & v_n \end{pmatrix} \tag{4.4.1}$$

(4.4a) Write an *efficient* MATLAB function `x = solveband(u,v,w,b)`, which solves the linear system of equations $\mathbf{Ax} = \mathbf{b}$ with Gaussian elimination without pivoting, and exploits its special structure.

(4.4b) Write a MATLAB function `A = MakeBand(u,v,w)` that computes a matrix $\mathbf{A}$ of the form (4.4.1) using the MATLAB `diag` command. Write another MATLAB routine `A = MakeBandSparse(u,v,w)` that computes the same matrix $\mathbf{A}$ exploiting the sparsity pattern of $\mathbf{A}$. Use the MATLAB command `spdiags`.

(4.4c) Check whether your implementations in subproblem (4.4a) and subproblem (4.4b) are correct by comparing your solution with the result returned by the standard MATLAB backslash solver for a few examples with strictly diagonally dominant matrices $\mathbf{A}$ of the form (4.4.1). For example, `u = ones(n-2,1)`, `v = -4*ones(n,1)`, `w = ones(n-1,1)` and `b = rand(n,1)` for a fixed `n`.

(4.4d) What is the asymptotic computational effort of your algorithm in subproblem (4.4a) in terms of the matrix size $n \to \infty$? Make a runtime comparison of `solveband` with the standard MATLAB solver for the case of $\mathbf{A}$ built as a full or as a sparse matrix using the routines in subproblem (4.4b). Assume $\mathbf{u} = (\frac{1}{2}, \dots, \frac{1}{2})^\top$, $\mathbf{w} = (\frac{1}{2}, \dots, \frac{1}{2})^\top$, $v_i = i$ for $i = 1, \dots, n$, $\mathbf{b} = (1, \dots, 1)^\top$ and for $n = 2^2, \dots, 2^{11}$. What are the empirically observed complexities of the different methods? Visualize the results by using double-logarithmic plots.

HINT: Use the MATLAB runtime commands `tic` and `toc`.

(4.4e) Now consider an $n \times n$ matrix $\mathbf{A}$ of the form (4.4.1) with $n = 50$ and compute its LU-decomposition using the standard MATLAB command `[L,U,P] = lu(A)`. Compare the structure of nonzero entries of the LU-factors $\mathbf{L}$ and $\mathbf{U}$ for the following choices

1. $v_i = -4$, $u_i = 1$ and $w_i = 2$ for all i,
2. the same entries as above except for $u_1 = u_3 = u_7 = u_{13} = u_{23} = u_{31} = u_{47} = 10$.

Explain the difference. Compute the bandwidth of $\mathbf{L}$ and $\mathbf{U}$.

HINT: use the `spy` command to explore the pattern of nonzeros of matrices.

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MATLAB: Submit all files in the online system. Include the files that generate the plots. Label all your plots. Include commands to run your functions. Comment on your results.

Written exercises: Hand-in the solutions during the exercise class or in the labeled boxes in HG G 53.x.

References

[NMI] Lecture Slides for the course "Numerical Analysis I".

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