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Numerical Analysis I

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D-MATH

Homework Problem Sheet 7

Introduction. This problem sheet is devoted to polynomial interpolation.

Problem 7.1 Nodal Polynomial in Interpolation

This problem looks at a few technical details connected with polynomial interpolation in order to practice manipulations and estimates.

(7.1a) Let $p_n \in \mathbb{P}_n$ be the interpolating polynomial of degree at most n for the data points $\{(x_i, y_i)\}_{i=0}^n \subset \mathbb{R}^2$ with $x_i = x_j \Rightarrow i = j$, so in particular, we have $p_n(x_i) = y_i$ for $i = 0, \dots, n$. Show that p_n is given by

$$p_n(t) = \sum_{i=0}^{n} \frac{\omega_{n+1}(t) y_i}{(t - x_i)\omega'_{n+1}(x_i)}, \quad \text{where} \quad \omega_{n+1}(t) = \prod_{i=0}^{n} (t - x_i). \tag{7.1.1}$$

(7.1b) Prove that for the *equidistant* nodes $-1 = x_0 < x_1 < \ldots < x_{n-1} < x_n = 1$ and n even, it holds

$$(n-1)! h^{n-1} |(t - x_{n-1})(t - x_n)| \leq |\omega_{n+1}(t)| \leq n! h^{n-1} |(t - x_{n-1})(t - x_n)|$$

for all $t \in (x_{n-1}, x_n)$ and $h = 2/n$.

Problem 7.2 Hermite Interpolation

Polynomial interpolation as treated in the course relied on providing pairs of distinct nodes x_j, $j = 0, \dots, n$, and point values y_j, $j = 0, \dots, n$. A generalization, known as *Hermite interpolation* prescribes *derivative values* (slopes) c_j, $j = 0, \dots, n$ in the interpolation nodes beside point values, that is we seek a polynomial p of suitable degree such that

$$p(x_j) = y_j \quad , \quad p'(x_j) = c_j \ , \quad j = 0, \dots, n \ . \tag{7.2.1}$$

In this problem we will see that this generalized interpolation problem is well posed.

Throughout we fix $m \in \mathbb{N}$ and the set $\mathcal{N} := \{x_0, \dots, x_m\} \subset \mathbb{R}$ of distinct nodes.

(7.2a) First, we look for counterparts of the Lagrange polynomials of classical polynomial interpolation. The task is to find $L_j^{(0)}$ and $L_j^{(1)}$ in $\mathbb{P}_{2m}$ such that for $i,\ j \in \{0, ..., m\}$,

$$L_j^{(0)}(x_i) := \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases} \quad , \quad (L_j^{(0)})'(x_i) = 0, \tag{7.2.2}$$

$$L_j^{(1)}(x_i) = 0 \quad , \quad (L_j^{(1)})'(x_i) := \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases} \quad . \tag{7.2.3}$$

(7.2b) Next, we tackle the well-posedness of the Hermite interpolation problem: Prove that for given $c_j,\ y_j,\ j = 0, ..., m$, there is a unique $q \in \mathbb{P}_{2m}$ such that $q(x_j) = y_j$ and $q'(x_j) = c_j$.

(7.2c) Surprisingly, in general it is not possible to prescribe point values and derivatives for polynomials at will: Show that there is no $p \in \mathbb{P}_3$ such that

$$p(-1) = 1,\ p'(-1) = 1,\ p'(1) = 2,\ p(2) = 1.$$

Problem 7.3 Integral Representation of Interpolation Error

In [NMI, Thm. 3.6] we found a representation for the error of polynomial interpolation of a function that relied on evaluating a derivative of the function at an unknown position (x_* in the statement of the theorem).

There is another family of error representation formulas for polynomial interpolation on an interval $[a, b]$ that are of the form ($f \in C^{n+1}([a, b])$)

$$(f - P_{\mathcal{N}} f)(t) = \int_a^b G_{\mathcal{N}}(t, \xi) f^{(n+1)}(\xi)\, \mathrm{d}\xi\ , \tag{7.3.1}$$

where $G_{\mathcal{N}} : [a, b]^2 \to \mathbb{R}$ is a suitable *kernel function*. In this problem we derive such a representation for the simple case of linear interpolation and use it for estimating the interpolation error.

Assume that $f \in C^2[a, b]$, $p \in \mathbb{P}_1$, defined by $p(a) = f(a),\ p(b) = f(b)$.

(7.3a) Show that

$$(p - f)(t) = \int_a^b G(t, \xi) f''(\xi) d\xi, \tag{7.3.2}$$

where the kernel function if given by

$$G(t, \xi) = \begin{cases} \dfrac{(b-t)(\xi-a)}{b-a} & a \le \xi < t \\ \dfrac{(t-a)(b-\xi)}{b-a} & t \le \xi \le b \end{cases} \quad . \tag{7.3.3}$$

HINT: Integrate by parts twice.

(7.3b) Error representations according to (7.3.1) are very useful for obtaining error estimates in norms that involve integrals.

Use Equation 7.3.2 to show that

$$\|f - p\|_{L^2([a,b])} \le (b-a)^2 \|f''\|_{L^2([a,b])}.$$

HINT: The L^2-norm of a continuous function g on $[a, b]$ is defined by

$$\|g\|^2_{L^2([a,b])} := \int_a^b |g(\xi)|^2 d\xi.$$

(7.3c) By differentiating Equation 7.3.2, show that

$$\|f' - p'\|_{L^2([a,b])} \le (b-a) \|f''\|_{L^2([a,b])}.$$

(7.3d) Error representations like (7.3.1) also yield estimates in the maximum norm, though they may not be a sharp as those extracted from [NMI, Eq. (3.11)].

Show that $f(a) = f(b) = 0$ implies that

$$\|f\|_{\infty,[a,b]} \leq (b-a)^2 \|f''\|_{\infty,[a,b]}.$$

Problem 7.4 Trigonometric Interpolation

Fourier sums and polynomials are closely related as has become apparent in the proof of [NMI, Thm. 3.23]. In this problem we study interpolation by Fourier sums that are aptly called *trigonometric polynomials*. Hence, the title of this problem.

Let $f \in C^0(\mathbb{R})$ be a 2π-periodic function, that is $f(t + 2\pi) = f(t)$ for all $t \in \mathbb{R}$. Consider the interpolation nodes $x_j = 2\pi j/n$ for $j = 0, \ldots, n-1$ and $n = 2m+1$ with $m \in \mathbb{N}$.

(7.4a) Show that there exists a unique vector $\mathbf{c} = (c_{-m}, \ldots, c_m) \in \mathbb{C}^n$ such that

$$q(x_j) = f(x_j) \quad \text{for } j = 0, \ldots, n-1 \quad \text{where} \quad q(t) = \sum_{k=-m}^{m} c_k \mathrm{e}^{\mathrm{i}kt}$$

HINT: Reduce to polynomial interpolation.

(7.4b) What is the expression of the interpolant $q(t)$ from subproblem (7.4a) when $f(t) = \mathrm{e}^{\mathrm{i}\ell t}$ and $\ell \in \mathbb{Z}$?

(7.4c) Let $f(t)$ be given as a Fourier series

$$f(t) = \sum_{\ell=-\infty}^{\infty} \widehat{f}_\ell \mathrm{e}^{\mathrm{i}\ell t} \quad , \quad \widehat{f}_\ell \in \mathbb{C} \, , \tag{7.4.1}$$

with $\left|\widehat{f}_\ell\right| \leq C\ell^{-2}$ for some $C > 0$ (Why this assumption?). Compute the corresponding trigonometric interpolant $q(t)$ as introduced in subproblem (7.4a).

HINT: Use subproblem (7.4b).

(7.4d) Finally, we tackle an interpolation error estimate for trigonometric interpolation based on the Fourier series representation (7.4.1) of the interpoland.

Find an estimate for the maximum norm of the interpolation error $\|f - q\|_{\infty,\mathbb{R}}$ when f and q are defined as in subproblem (7.4c) and $\left|\widehat{f}_\ell\right| \leq C\ell^{-r}$, $r \in \mathbb{N} \setminus \{1\}$, for some $C > 0$. Discuss what plays the role of the smoothness assumptions on the interpoland necessary for getting decay rates for the interpolation error.

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MATLAB: Submit all files in the online system. Include the files that generate the plots. Label all your plots. Include commands to run your functions. Comment on your results.

Written exercises: Hand-in the solutions during the exercise class or in the labeled boxes in HG G 53.x.

References

[NMI] Lecture Slides for the course "Numerical Analysis I".

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