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Numerical Analysis I

ETH Zürich
D-MATH

Homework Problem Sheet 11

Introduction. This problem sheet is devoted to iterative methods for non-linear (systems of) equations, in particular Newton's Method.

Problem 11.1 Computing an Important Function using Newton's Method

(11.1a) In this problem you are confronted with a task of "reverse engineering" and you will be asked to explain an undocumented MATLAB code. Unfortunately, one is often confronted with such challenges, because programmers of numerical software may not have documented their codes properly.

Explain the following MATLAB code line by line. What is its purpose?

```
function y = myfn(x)
log2 = 0.693147180559945;
y = 0;
while (x > sqrt(2))
    x = x/2; y = y + log2;
end
while (x < 1/sqrt(2))
    x = x*2; y = y - log2;
end
z = x-1;
dz = x*exp(-z)-1;
while (abs(dz/z) > eps)
    z = z+dz;
    dz = x*exp(-z)-1;
end
y = y+z+dz;
```

Problem 11.2 MATLAB: Newton's Method for the Eigenvalue Problem

Let $\mathbf{A} \in \mathbb{R}^{n \times n}$ have only real eigenvalues and eigenvectors. We are looking for a pair of eigenvector and eigenvalue $(\mathbf{v}, \lambda) \in \mathbb{R}^n \times \mathbb{R}$ of $\mathbf{A}$ that satisfies the equation $\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$ under the constraint that $\mathbf{v}^\top\mathbf{v} = 1$.

(11.2a) Rewrite the problem in the form $\mathbf{F}(\mathbf{v}, \lambda) = \mathbf{0}$, where $\mathbf{F} : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^{n+1}$.

(11.2b) Formulate Newton's method for the non-linear system of equations $\mathbf{F}(\mathbf{c}, \lambda) = 0$.

(11.2c) Implement a MATLAB function

```
[l, T] = fixpkt(phi, v, l, tol, maxit)
```

that performs a fixed point iteration on the pair $(\mathbf{v}, \lambda)^\top$ and takes as input the Newton iteration function `phi` found in subproblem (11.2b), the initial guess (`v`, `l`), the maximum number of iterations `maxit` and the tolerance `tol`. As stopping criterion, use the distance between two consecutive approximations for the eigenvalue. Intermediate values of the eigenvalues have to be stored in the output vector `T` except for the last one `l`.

Test your code for:

```
m = 5; n = m^2;
A = gallery('poisson',m);
v = 1/2/n*ones(n,1); l = 0;
[l, T] = fixpkt(phi, v, l, 1e-10, 25)
```

and a suitable implementation of `phi`.

Listing 11.1: Test call output for subproblem (11.2c)

```
l = 2.2679

T =       0    30.2953    14.8882     6.7604     2.3736     1.1300
     2.2555     2.2669     2.2676     2.2679     2.2679     2.2679
```

(11.2d) Using the function `fixpkt` from subproblem (11.2c), determine (numerically) the local convergence rate of the Newton iteration from subproblem (11.2b) for the following choice of the matrix $\mathbf{A} = \mathbf{A}_i$, $i \in \{1, 2, 3\}$, where

$$\mathbf{A}_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{A}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 10 \end{pmatrix}, \quad \mathbf{A}_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Use as initial guess $\mathbf{v}^{(0)} = (-1, 1)^\top$, $\lambda^{(0)} = 0$ and display in a semilog plot the deviation $|\lambda_k - \lambda_K|$, $0 \leq k < K$. In this case $K \geq 0$ is minimal, in such a way that either $|\lambda_{K-1} - \lambda_K| \leq$ `tol` $= 10^{-14}$ or $K \geq$ `maxit` $= 25$ holds. What can you infer?

(11.2e) Prove what you have numerically observed in subproblem (11.2d): The Newton iteration from subproblem (11.2b) converges locally quadratically if λ has algebraic multiplicity *one*.

HINT: Look at [NMI, Thm. 5.20]. Consider the geometric and algebraic multiplicity of λ and distinguish among cases.

Problem 11.3 Convergence of Newton's Method

Prove the following result.

Theorem. Let $D \subset \mathbb{R}^n$ be open and convex, and let $F : D \mapsto \mathbb{R}^n$ continuously differentiable. Suppose that there exists $\mathbf{x}^* \in D$ with $F(\mathbf{x}^*) = 0$ such that $DF(\mathbf{x}^*)$ is regular. Moreover, assume there holds the affine invariant Lipschitz condition

$$\exists L > 0 : \quad \left\| DF(\mathbf{x}^*)^{-1}(DF(\mathbf{y}) - DF(\mathbf{x})) \right\|_2 \leq L\|\mathbf{y} - \mathbf{x}\| \quad \forall \mathbf{x}, \mathbf{y} \in D \,, \qquad (11.3.1)$$

and

$$\rho := \left\|\mathbf{x}^{(0)} - \mathbf{x}^*\right\|_2 < \frac{2}{3L} \quad \wedge \quad B_\rho(\mathbf{x}^*) \subset D \; .$$

Then the Newton iteration for solving $F(\mathbf{x}) = 0$ with initial guess $\mathbf{x}^{(0)}$ satisfies

(i) $\mathbf{x}^{(k)} \subset B_\rho(\mathbf{x}^*)$ for all $k \in \mathbb{N}_0$,

(ii) $\mathbf{x}^{(k)} \to \mathbf{x}^*$ for $k \to \infty$.

HINT: Get clues from the proof of Theorem 5.20 (NMI_5) and apply to the Jacobi matrix $DF(\mathbf{x})$ the *Perturbation bound in matrix inversion* in the form: Suppose that $\mathbf{A}$ and $\mathbf{B}$ are two matrices in $\mathbb{R}^{n\times n}$, and $\mathbf{A}$ is regular. In addition, suppose that for some real α and κ, $\|\mathbf{A}^{-1}\| \leq \alpha$ and $\|\mathbf{A}^{-1}\|\|\mathbf{B} - \mathbf{A}\| \leq \kappa < 1$. Then $\mathbf{B}$ is regular and $\|\mathbf{B}^{-1}\| \leq \frac{\alpha}{1-\kappa}$.

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MATLAB: Submit all files in the online system. Include the files that generate the plots. Label all your plots. Include commands to run your functions. Comment on your results.

Written exercises: Hand-in the solutions during the exercise class or in the labeled boxes in HG G 53.x.

References

[NMI] Lecture Notes for the course "Numerical Analysis I".

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