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Numerical Methods for Partial Differential Equations

ETH Zürich
D-MATH

Homework Problem Sheet 1

Introduction. All the problems are devoted to study some general concepts of partial differential equations considering the one-dimensional Poisson equation as model problem.

Problem 1.1 Green's function for Poisson equation

We consider the 1d Poisson equation with homogeneous Dirichlet boundary conditions:

$$\begin{aligned} -u''(x) &= f(x), \quad \forall x \in \Omega = (0,1) \\ u(0) &= u(1) = 0. \end{aligned} \tag{1.1.1}$$

In the lecture, we learned that the solution u to this equation can be obtained using the Green's function:

$$u(x) = \int_\Omega G(x,y) f(y)\,\mathrm{d}y, \quad x \in \Omega$$

(in case of nonhomogeneous boundary conditions, we would have also bounday terms).
The Green's function for this problem is defined as:

$$G_x(y) := G(x,y) = \begin{cases} y(1-x) & \text{for } 0 \le y \le x \\ x(1-y) & \text{for } x \le y \le 1. \end{cases} \tag{1.1.2}$$

Its expression was obtained by simple integration by parts of (1.1.1).

However, Green's function can be introduced in a more general way for any PDE.
Before this, we need to recall the concept of Dirac delta distribution.
The delta distribution $\delta_x(y) := \delta(x-y)$ centered at the point y can be thought, at first glance (rigorous definition goes beyond the scope of this course), as a function

$$\delta(x-y) = \begin{cases} 0 & \text{for } y \neq x \\ \infty & \text{for } y = x. \end{cases} \tag{1.1.3}$$

Physically, it describes an impulse (e.g. the force excited when you hit strongly a table with an hammer).
The delta distribution has the property of "selecting" function values:

$$\int_\Omega \delta(x-y) f(y)\,\mathrm{d}y = f(x) \quad \text{for any } f \in C([0,1]) \tag{1.1.4}$$

(from which $\int_\Omega \delta(x-y)\,\mathrm{d}y = 1$ follows).
Then the Green's function $G(x,y) := G_x(y)$ for a PDE is defined as the impulse response, i.e.

as the solution to the PDE when at the right-handside we have $f(x) = \delta_x(y)$. For the Poisson's equation, this means:

$$\begin{aligned} -G''_x(y) &= \delta_x(y), \quad \forall y \in \Omega = (0,1) \\ G_x(0) &= G_x(1) = 0. \end{aligned} \tag{1.1.5}$$

(1.1a) Show that the Green's function as given by (1.1.2) solves (1.1.5).

HINT: When a function $g(y)$ has a jump at a point x, then, it is not differentiable in that point. However, a generalized derivative can be defined, so that $g'(y)|_{y=x} = \delta_x(y)|_{y=x}$.
Consequently, the derivative of a piecewise constant function $g(y)$ with jump at x is $g'(y) = \delta_x(y)$.

(1.1b) Show that the function u defined as

$$u(x) = \int_0^1 G(x,y) f(y)\,\mathrm{d}y, \quad x \in (0,1), \tag{1.1.6}$$

satisfies (1.1.1).

HINT: Formula (1.1.4) may be useful.

(1.1c) Implement a function

```
function val = Green(x,y)
```

which accepts in input two column arrays `x` and `y` of grid points, and returns in the matrix `val` the Green's function (1.1.2) evaluated at those grid points, with the convention that `val(i,j)=Green(x(i),y(j))`.

(1.1d) Use the routine of subproblem (1.1c) to plot the Green's function for $x = \frac{1}{4}, \frac{1}{2}, \frac{3}{4}$ ($y \in [0,1]$).

(1.1e) Implement a function

```
function u = PoissonGreen(x,FHandle)
```

which accepts as input a column vector `x` of equispaced grid points $\{x_0 = 0, x_1, \dots, x_N, x_{N+1} = 1\}$; and a function handle `FHandle` to the right-handside f of (1.1.1); in output it returns a column vector `u` containing the value of the solution u to (1.1.1) computed at the points `x`. The solution is computed using the formula (1.1.2).
To compute the integral, use the composite trapezoidal quadrature rule:

$$\int_0^1 g(x)\,\mathrm{d}x \approx \frac{g(0)h}{2} + h \sum_{i=1}^{N} g(x_i) + \frac{g(1)h}{2}, \tag{1.1.7}$$

where $h = |x_1 - x_0|$.
Use the array `x` itself as quadrature points.

HINT: Use the implementation from subproblem (1.1c) for the Green's function.

We are now going to discretize (1.1.1) using the Finite Differences method, namely the centered finite differences.
To this aim, we subdivide the interval $[0, 1]$ in $N + 1$ subintervals using equispaced grid points $\{x_0 = 0, x_1, \ldots, x_N, x_{N+1} = 1\}$.
The discretized problem can be written as a linear system

$$\boldsymbol{A}\boldsymbol{u} = \boldsymbol{L}, \tag{1.1.8}$$

where $\boldsymbol{A}$ is a $N \times N$ matrix, $\boldsymbol{L}$ a $N \times 1$ vector and $\boldsymbol{u}$ the $N \times 1$ vector containing of unknowns $u(x_j)$, $j = 1, \ldots, N$, the value of the function u at the grid points.
Let us denote by $h = |x_1 - x_0|$ the meshsize.

(1.1f) Refresh your mind on what we saw in class on the Finite Difference scheme and in particular on the central finite differences.
Write the matrix A and the right-handside L. For the right-handside, write it in terms of a generic force term $f(x)$ in (1.1.1) when the composite trapezoidal quadrature rule (1.1.7) is applied.

(1.1g) Implement a function

```
function A = PoissonMatrix(N)
```

which computes the matrix $\boldsymbol{A}$ for (1.1.8). Here the input parameter `N` denotes the number of *internal* grid points.

(1.1h) Implement a function

```
function L = PoissonRHS(FHandle,N)
```

to compute the right-handside $\boldsymbol{L}$ for (1.1.8). The input parameter `FHandle` is the function handle for the right-handside $f(x)$ and `N` is again the number of interior grid points.

(1.1i) Implement the function

```
function uh = PoissonSolve(FHandle,N)
```

to solve the Poisson problem (1.1.1).
The input parameters are as in subproblem (1.1h). The output `uh` is the array $\{u_h(x_j)\}_{j=1}^N$ containing the approximate value of the solution u at the interior grid points $\{x_j\}_{j=1}^N$.

HINT: Use the routines from subproblems (1.1g) and (1.1h).

(1.1j) Run the routine `PoissonSolve` for $f(x) = \sin(2\pi x)$ and $N = 50$ and plot the solution.

We saw in the lecture that the centered finite difference schemes satisfies is stable and consistent, and thus it converges to the exact solution u to (1.1.1) when the mesh is refined.
Here we are going to test the convergence of our scheme through a convergence study.

(1.1k) Write a function

```
function [h err] = Poissoncvg(FHandle)
```

to perform the convergence study. The input argument is a function handle to the right-hanside $f(x)$. As output the routine returns the array h=$[h_1, \ldots, h_6]$ of meshsizes and the array err=$[e_1, \ldots, e_6]$ of computed errors.
As error between the discrete solution u_h and the exact solution u, we consider the maximum norm error $\|u - u_{h_i}\|_\infty = \max_{x\in[0,1]} |u - u_h|$, $i = 1, \ldots, 6$; we can compute this error just in an approximate way: we consider a very fine grid with meshsize $h_{ref} = \frac{1}{2^{10}}$ and grid points $x_0 = 0, x_1 = h_{ref}, \ldots, x_{N+1} = 1$ and approximate the maximum norm by

$$\|u - u_h\|_\infty \approx \max_{x_0 \ldots x_{N+1}} |u(x_i) - u_h(x_i)| \tag{1.1.9}$$

The standard steps for a convergence study then:

1. compute the *reference solution*: compute the exact solution u to (1.1.1) that you can obtain by call to the routine PoissonGreen from subproblem (1.1e) on the grid with meshsize h_{ref};
2. start from a meshsize $h_1 = \frac{1}{4}$, corresponding to $N_1 = 3$ interior grid points;
3. compute the discrete solution u_{h_1} to (1.1.1);
4. compute the error $e_1 \approx \|u - u_{h_1}\|_\infty$; inside each mesh interval, consider the linear interpolant for u_h;
5. refine the grid, considering $h_2 = \frac{h_1}{2} = \frac{1}{8}$ and repeat the algorithm from step 3;
6. repeat the previous step till $h_6 = \frac{h_1}{2^5}$.

(1.1l) Run the routine Poissoncvg for $f(x) = \sin(2\pi x)$.
Make a double logarithmic plot of the errors $e_1, \ldots, e_6$ versus the meshsizes $h_1, \ldots, h_6$. What do you observe? Which is the order of convergence?

Problem 1.2 The Poisson equation with Neumann boundary conditions

We consider the Poisson equation with homogeneous Neumann boundary conditions:

$$\begin{aligned} -u''(x) &= f(x), \quad \forall x \in \Omega = (0,1) \\ u'(0) &= u'(1) = 0, \end{aligned} \tag{1.2.1}$$

$f \in C^0([0,1])$.

(1.2a) Show that

$$\int_0^1 f(x)\,\mathrm{d}x = 0 \tag{1.2.2}$$

is a necessary condition to have a solution.

(1.2b) Show that (1.2.1) does not have a unique solution.

HINT: Suppose that a function $u(x)$ is a solution and consider $v(x) = u(x) + c$, $c > 0$.

(1.2c) Show that, if (1.2.2) is fulfilled, then (1.2.1) has always a solution $u \in C^2([0,1])$. Moreover, show that the solution is unique if we require

$$\int_0^1 u(x)\,\mathrm{d}x = 0 \tag{1.2.3}$$

We are now going to analyse the discretization (1.2.1) through centered differences.
Consider a partition $\{x_0 = 0, x_1, \ldots, x_N, x_{N+1} = 1\}$ of the interval $[0,1]$, with equispaced points and meshsize h. The discretized equation can be written as

$$\boldsymbol{A}\boldsymbol{u} = \boldsymbol{L}, \tag{1.2.4}$$

where $\boldsymbol{A}$ is a $(N+2) \times (N+2)$ matrix, $\boldsymbol{L}$ is a vector of length $N+2$ and $\boldsymbol{u}$ is the vector of unknowns $\{u_h(x_0), \ldots u_h(x_{N+1})\}$, where u_h denotes the discrete solution.

(1.2d) Compute the matrix $\boldsymbol{A}$ and the right-handside $\boldsymbol{L}$.

(1.2e) Show that the matrix $\boldsymbol{A}$ is singular. This is because the nonunicity of the solution that we have seen in subproblem (1.2b) is reflected on the discrete case.

(1.2f) We have seen in subproblem (1.2c) that under the constraint (1.2.3) the solution to (1.2.1) is unique.
Apply the composite trapezoidal quadrature rule to (1.2.3) to get an equation in terms of the entries of $\boldsymbol{u}$, i.e. a constraint on the discrete level.

HINT: The composite trapezoidal quadrature rule for a generic function g is:

$$\int_0^1 g(x)\,\mathrm{d}x \approx \frac{g(0)h}{2} + h\sum_{i=1}^{N} g(x_i) + \frac{g(1)h}{2}, \tag{1.2.5}$$

(1.2g) Combining the equation we have got from subproblem (1.2f) with the system of equations (1.2.4), we have a linear system of $N+3$ equations for $N+2$ unknowns.
However, these equations are not linearly independent since the rows of $\boldsymbol{A}$ are not. It can be shown (and actually it is expected from subproblem (1.2c)) that the equation we get from subproblem (1.2f) is linearly independent from the rows of $\boldsymbol{A}$. Then, summing up this equation with the first equation in (1.2.4), we get a system

$$\tilde{\boldsymbol{A}}\boldsymbol{u} = \tilde{\boldsymbol{L}} \tag{1.2.6}$$

of $N+2$ linearly independent equations for $N+2$ unknowns.
Write the matrix $\tilde{\boldsymbol{A}}$ and the vector $\tilde{\boldsymbol{L}}$ (the latter in terms of a generic right-handside $f(x)$).

(1.2h) Write a routine

```
function A = PoissonNeuMatrix(N)
```

to implement the matrix $\tilde{\boldsymbol{A}}$. Here N is the number of *interior* grid points.

(1.2i) Write a routine

```
function L = PoissonNeuRHS(FHandle,N)
```

to implement the right-handside $\tilde{\boldsymbol{L}}$. Here `FHandle` is a function handle to a generic right-handside $f(x)$ and N is the number of *interior* grid points.

(1.2j) Write a routine

```
function uh = PoissonNeuSolve(FHandle,N)
```

to solve the system (1.2.6) and store the solution in the vector `uh`.

(1.2k) Consider $f(x) = \sin(2\pi x)$.
Verify that this right-handside satisfies the condition (1.2.2).
Compute the analytic solution for this right-handside.

(1.2l) Use the routine `PoissonNeuSolve` to compute the solution to (1.2.1) with $f(x) = \sin(2\pi x)$. Use $N = 50$ and make a plot of the solution.
You can compare then this plot with a plot of the exact solution computed in (1.2k) to have a first check of your routines (although an exhaustive check would require a convergence study).

Problem 1.3 Stability property for the Poisson equation

The aim of this problem is to better understand what does the *stability* mean for a differential equation. We will do it consider the Poisson equation with homogeneous Dirichlet boundary conditions:

$$\begin{aligned} -u''(x) &= f(x), \quad \forall x \in \Omega = (0,1) \\ u(0) &= u(1) = 0. \end{aligned} \tag{1.3.1}$$

for $f \in C([0,1])$.

In applications the exact source term $f(x)$ is not available. What is available is some perturbation of it

$$\tilde{f}(x) = f(x) + \eta(x), \tag{1.3.2}$$

where $\eta(x)$ is some noise introduced, for example, by some measurement error.
Then, what can be actually computed is the solution $\tilde{u}$ to the perturbed system

$$\begin{aligned} -\tilde{u}''(x) &= \tilde{f}(x), \quad \forall x \in \Omega = (0,1) \\ \tilde{u}(0) &= \tilde{u}(1) = 0. \end{aligned} \tag{1.3.3}$$

(1.3a) Show that

$$\|u - \tilde{u}\|_\infty \le \frac{1}{8}\|f - \tilde{f}\|_\infty \tag{1.3.4}$$

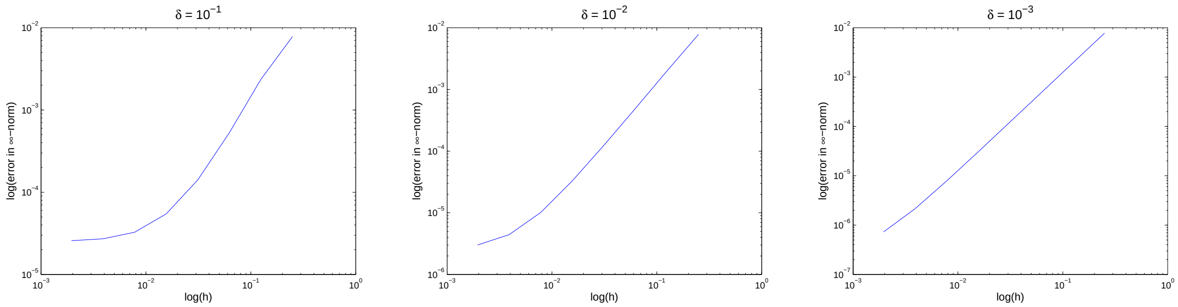

Figure 1.1: Plots for subproblem (1.3c).

(1.3b) Consider now that we want to compute the solution to (1.3.3) numerically and denote by $\tilde{u}_h$ the discrete solution. We are interested in estimating how well $\tilde{u}_h$ approximates the exact solution u to the *unperturbed* problem (1.3.3).
Show that the following estimate holds:

$$\|u - \tilde{u}_h\|_\infty \leq \frac{1}{8}\|\eta\|_\infty + \|\tilde{u} - \tilde{u}_h\|_\infty. \tag{1.3.5}$$

Note that here we are not making any assumption on the discretization scheme.

(1.3c) Suppose that $\eta(x) = \delta\sin(2\pi x)$, for some $\delta > 0$, so that

$$\tilde{f} = f + \delta\sin(20\pi x). \tag{1.3.6}$$

Let us consider $\delta = 10^{-1}, 10^{-2}, 10^{-3}$ and suppose that for each of this values we made a convergence study for $\tilde{u}_h$ considering the error $\|u - \tilde{u}_h\|_\infty$. The convergence plots are shown in Fig. 1.1 Observe and compare the plots and comment on them:

- Why, for meshsize h small there is a plateau?
- How does the plateau change with δ? Why?

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