

CHAPTER 1: INTRODUCTION

1. What are PDE's:

As the name suggests, Partial Differential Equations or PDEs are equations that involves partial derivatives.

To be more precise, consider a domain $\Omega \subseteq \mathbb{R}^n$. Here, $n=1,2,3$ is the dimension and Ω is an open set: (see figure 1)

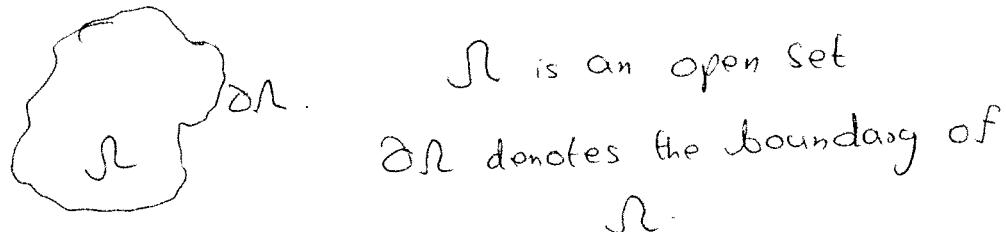


Figure 1.

Then, the unknown in any PDE is a function U defined as:

$U: \Omega \mapsto \mathbb{R}^m$ i.e, for any $x \in \Omega$, the function U provides a unique value $U(x)$ that is an m -vector. Note that for $m=1$, $U(x)$ is a scalar.

If U is a smooth function, then the derivatives of U are well defined.

Denote ∇U as the gradient of U , D^2U as the second derivative and so on,

Then a PDE is a relation of the form:

$$F(U, \nabla U, D^2U, \dots, D^m U, \dots, D^k U) = S - \textcircled{i}$$

The above relation holds at any point $x \in \Omega$.

NB: F can be a nonlinear function.

Thus, given a PDE of the form ①, we have to find
the unknown U from the complicated non-linear relationship
between a combination of its partial derivatives. (2)

NB: Definition: Order of the PDE ① \Rightarrow : Order of the PDE is the highest derivative associated contained in the PDE.

NB: we will restrict ourselves mostly to first-order and second-order PDEs in this course.

2: Why study PDEs: Most, if not all, phenomena of interest in Physics, Chemistry, Engineering, Biology and increasingly in the economic and social sciences, are modeled (mathematically) in terms of PDEs.

Some examples of such models are in:

1. Weather forecasting:

PDEs involved are: Euler equations, Navier-Stokes equations
Shallow-Water equations

2. Tsunami/floods prediction:

: Shallow Water equations

3. Design of Aerospace Vehicles (Aircraft, Missiles, Engines)

: Euler equations, Navier-Stokes equations

4. Design of Wave guides, Radars, Stealth fighters:

: Maxwell's equations.

(S)

5. Astrophysics: Solar Atmosphere, Super Novas, Exoplanets

: Magneto hydrodynamics (MHD) Equations.

6. Quantum physics: Lasers, Quantum computing, Nano-objets:

: Schrödinger Equations.

7. Design of Structures, Smart materials

: Elasticity equations

8. Developmental Biology:

: Reaction-Diffusion Equations.

9. Economics - Supply chain management, Social dynamics

: Kinetic equations

10. finance - Option pricing

: Black-Scholes equations.

and many many more.

Or as Herbert Simon (Nobel laureate in Economics) said

“

3. How do PDE's arise?

(4)

These are two main ways in which PDE's arise in models.

A. Conservation (Balance) Laws:

Let U denote some quantity of interest - say concentration of a chemical, pressure, temperature or density of a gas and so on. Then we have a fundamental conservation law:

(CL) The rate of change (in time) of U (in any subdomain $\omega \subseteq \mathbb{R}^n$) is given by the flux (the amount of U entering or leaving) through the boundary $\partial\omega$ of ω and the source (sink) i.e. the amount of U produced or destroyed inside ω .

To phrase the above statement mathematically;

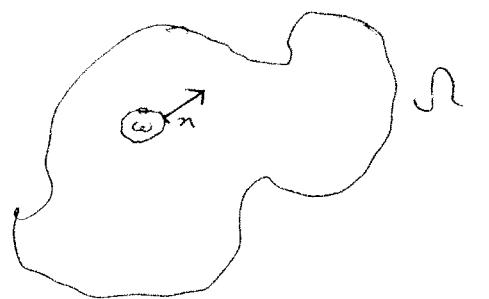
$$\text{Total amount of } U \text{ inside } \omega = \int_{\omega} u \, dx$$

$$\text{Rate of change of } U = \frac{d}{dt} \int_{\omega} u \, dx$$

Flux through the boundary at any point = F

$$\text{Source Total flux through the boundary} = - \int_{\partial\omega} F \cdot n \, ds(x)$$

Here n is the unit outward normal to $\partial\omega$ see figure (2).



The source at each point inside ω = S

$$\text{Total Source} = \int_{\omega} S \, dx$$

(5)

Then (4) \Rightarrow

$$\frac{d}{dt} \int_{\omega} u dx = - \underbrace{\int_{\partial\omega} F \cdot n ds(x)}_{(\text{flux})} + \int_{\omega} s dx - ②$$

(rate of change) (source)

By Gauss divergence theorem:

$$\int_{\partial\omega} F \cdot n ds(x) = \int_{\omega} \operatorname{div}(F) dx$$

• Similarly if U is smooth:

$$\frac{d}{dt} \int_{\omega} u dx = \int_{\omega} U_t dx.$$

∴ From ②, we obtain:

$$\int_{\omega} (U_t + \operatorname{div} F - S) dx \equiv 0.$$

The above integral identity holds for all subdomains
 $\omega \subseteq \Omega$. Therefore,

$$U_t + \operatorname{div} F = S, \quad \forall x \in \Omega. \quad -③$$

• In practise, the flux F needs to be modeled and does the source S . Furthermore,

$$F = F(U, \nabla U, \underline{\Delta^2 U}).$$

$$S = S(U, \nabla U).$$

Hence, (3) is of the form:

(6)

$$U_t + \operatorname{div} f(u, vu) = S(u, vu) \quad - (3)$$

Eqn (3) is a prototypical form of a class of PDE's termed as conservation laws.

The flux f and source S need to be modeled from the specific domain where the model arises e.g. in Physics, Engineering or Biology. We provide some examples below.

Example 1: A simple 1-dimensional traffic flow model:

Consider a 1-D road with traffic consisting of similar types (homogeneous) cars.

The quantity of interest: $U = \rho$ (car number density) i.e. number of cars per unit length (assume that each car has negligible length,

Source $\equiv 0$ (no cars are created or destroyed on the road !!!)

flux $\equiv \rho V$: density \times Velocity (like a car momentum).

So the relevant PDE is:

$$\rho_t + (\rho V)_x = 0$$

The car macroscopic velocity V needs to be modeled. Lighthill, Whitham and Richards (LWR) used a very simple heuristics to model:

$$V := V_{\max}(1 - \rho) \quad - (4)$$

Here we normalize the car-carrying capacity of the road to be 1.

further, the model captures an essential fact: (7)

"If there are no (very few) cars on the road, $\rho \approx 0$ and each car can move at the maximum velocity i.e. speed limit"

"If there is a traffic jam, then $\rho \approx 1$ (maximum capacity) and the cars hardly move"

So, the subsequent PDE for traffic flow is:

$$\rho_t + \cancel{\rho_x} (v_{\max} \rho (1 - \rho))_x = 0 : - (5)$$

with the unknown $\rho(x, t)$ being the car density.

Remark: Note that the traffic flow ~~flow~~ (LWR) PDE encapsulates the essence of PDEs - ρ depends on two variables t and x and the PDE relates partial derivatives ρ_t and ρ_x .

Example 2: Euler equations of fluid dynamics:

A much more complicated example of ~~a~~ conservation laws are the equations of fluid dynamics, derived by Euler in 1770's.

Quantities of interest: fluid density ρ

fluid velocity $\mathbf{u} \cancel{t} = (u_1, u_2, u_3)$

pressure P

momentum $\rho \mathbf{u}$

Total energy E given by equation of state:

$$E := \frac{P}{\gamma - 1} + \frac{1}{2} \rho u^2 - \textcircled{6a}$$

internal energy kinetic energy

Here, γ is the gas constant.

(8)

Without explaining the details, the exact conservation laws are:

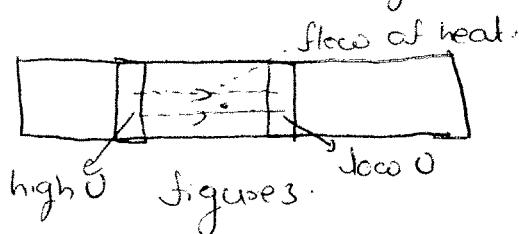
1. Conservation of mass: $\rho_t + \operatorname{div}(\rho u) = 0$ — (6a)
2. Conservation of momentum: $(\rho u)_t + \operatorname{div}(\rho u \otimes u + \rho I_D) = 0$ — (6b)
3. Conservation of energy: $E_t + \operatorname{div}((E+p)u) = 0$ — (6c)

Here I_D is the 3×3 identity matrix and $A \otimes B$ is the component-wise tensor product.

The Equations (6) are the celebrated Euler equations of gas-dynamics.

Example 3: The Heat Equation:

Consider a metal rod (see figure 3)



The quantity of interest in this model is the temperature (temperature density), still denoted by $U = U(x, t)$. Assume that the rod is heated. This heating can be modeled by some source function: $f(x, t)$.

In order to complete the conservation law (3), we need to define the flux F . To do so, one uses the Fourier's law:

"Heat flows from high temperature to low temperature". Hence, heat flux is proportional to the gradient of temperature:

i.e.

$$F = -k \nabla U$$

The minus sign represents the fact that heat flows in the direction of the negative gradient. The constant k is termed as the heat conductivity coefficient. For simplicity, let $k=1$. (9)

Combining the flux into the conservation law, we obtain a PDE modeling heat conduction.

$$u_t - \operatorname{div}(\nabla u) = f \\ \Rightarrow u_t - \Delta u = f \quad \text{--- (7)}$$

The Eqn (7) is the famous Heat equation.

Of particular interest is a steady state solution of (7) i.e.

$$u(x, t) \equiv u(x), \quad \forall t.$$

Applying the above ansatz into the heat equation (7), we obtain another famous equal PDE: the Poisson's equation:

$$-\Delta u = f \quad \text{--- (8)}$$

B. Variational principles:

In many situations in physics, the model boils down to choosing one configuration of a system among many possible configurations. The sought for configuration needs to be stable and is usually given as an optimizer (maximizes or minimizes) of a some energy function in an admissible class of functions.

We illustrate this general principle with an example:

Consider an elastic body with a configuration (state),
 defined on the domain Ω . The ~~domain~~ body (say a membrane) is
 clamped at the boundary of the domain - See figure 4. (10)

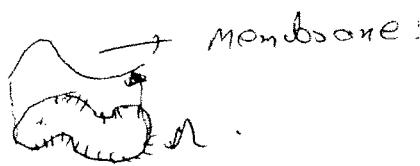


Figure 4.

The quantity of interest is the displacement u . The clamping at the boundary implies that

$$u|_{\partial\Omega} \equiv 0 \quad \text{--- (8)}$$

The total energy for any configuration u is given by:

$$J(u) := \frac{1}{2} \int_{\Omega} u u^2 dx + \int_{\Omega} u f dx \quad \begin{matrix} \downarrow \\ \text{kinetic energy} \end{matrix} \quad \begin{matrix} \downarrow \\ \text{potential energy} \end{matrix} \quad \text{--- (9)}$$

The sought for configuration is a minimizer of the total energy $J(u)$ among functions that satisfy the boundary condition (8).

The energy J is termed as the Dirichlet energy. To calculate the minimizer in (9), we need to ~~calculate~~ calculate the "so-called" Euler-Lagrange equations i.e.

$$0 = J'(u) = \lim_{\epsilon \rightarrow 0} \frac{J(u + \epsilon v) - J(u)}{\epsilon}$$

lets calculate;

$$J(u+cv) = \frac{1}{2} \int |\nabla u|^2 + c \int \langle \nabla u, \nabla v \rangle + \frac{c^2}{2} \int |\nabla v|^2$$

$$J(u) = \frac{1}{2} \int |\nabla u|^2 - \int uf + \int vf - c \int vf$$

$$\therefore \frac{J(u+cv) - J(u)}{c} := \int \langle \nabla u, \nabla v \rangle + \frac{c}{2} \int |\nabla v|^2 - c \int vf.$$

$$\lim_{c \rightarrow 0} \frac{J(u+cv) - J(u)}{c} = \underbrace{\int_N \langle \nabla u, \nabla v \rangle dx}_{I_1} + \underbrace{\int_N vf dx}_{I_2}$$

$$6 \quad J'(u) = \int_N \langle \nabla u, \nabla v \rangle dx - \int_N vf dx$$

by integrating by parts and using the boundary condition

$$u|_{\partial N} = 0, \text{ we obtain}$$

$$\int_N \langle \nabla u, \nabla v \rangle dx = - \int_N \Delta u v dx$$

\therefore The minimizer satisfies the Euler-Lagrange equations:

$$J'(u) \equiv 0$$

$$\Rightarrow \int (-\Delta u + f) v dx = 0$$

This holds for all test functions v and hence it holds pointwise i.e,

(12)

$$-\Delta u = f$$

The above is the Poisson's equation: (8).

Thus, the Poisson's equation can be derived from both a conservation law as well as a variational principle.

4. Types of PDEs.

There are 3 different classifications of PDEs. We will mention them and provide some examples below -

1. Linear vs. Non-linear PDEs.

Linear PDEs \rightarrow linear combination of solutions is a solution.

E.g. Poisson's equation, Heat equation, Wave equation.

Nonlinear PDEs \rightarrow everything else.

E.g. Traffic flow equation, Euler equations.

Most PDEs in application are Non-linear.

2. Elliptic PDEs — Poisson's equation

Parabolic PDEs — Heat Equation

Hyperbolic PDEs — Traffic flow equation, Euler equations.

3. Order of the PDEs.

First-order — Traffic flow, Euler equation.

Second-order — Poisson, Heat equation

5. Scope and contents of the course.

(B)

Given any PDE, we are interested in obtaining the solution. It is unrealistic to be able to find explicit solution formulas for particularly, nonlinear PDEs. Hence, we need suitable numerical algorithms to compute (approximate) solutions of PDE.

In this course, we will consider the following topics:

1. Finite difference methods for the Poisson's equation.
2. finite element methods for the Poisson's equation
3. finite difference and finite element methods for the Heat equation
4. finite difference methods for nonlinear reaction-diffusion equations.
5. finite volume methods for nonlinear hyperbolic equations

Chapter 2: The One-dimensional Poisson's Equation:

Given any domain $\Omega \subseteq \mathbb{R}^n$, the Poisson's equation is given by:

$$-\Delta u = f \quad \text{--- (1), } \forall x \in \Omega.$$

This equation needs to be augmented with boundary conditions. A common boundary condition, the so-called "Dirichlet" boundary condition is:

$$u|_{\partial\Omega} = 0.$$

The simplest form of the Poisson's equation is in one space dimension. In this case, the domain $\Omega = (0, 1)$ (without loss of generality) and the Poisson's equation (1) reduces to:

$$-u''(x) = f(x), \quad \forall x \in (0, 1) \quad \text{--- (2)}$$

With Dirichlet boundary conditions,

$$u(0) = u(1) = 0 \quad \text{--- (2)}$$

The problem (2-2) is an example of a 2-point boundary value problem (BVP).

1. Explicit solution formula through Green's function representation.

The two-point BVP can be solved explicitly in the following manner.

By fundamental theorem of integral calculus,

$$u'(y) = c_2 + \int_0^y u''(z) dz$$

$$= c_2 - \int_0^y f(z) dz \quad (\text{from the ODE } ②)$$

Another application of fundamental theorem of integral calculus yields:

$$u(x) = c_1 + \int_0^x u'(y) dy$$

$$= c_1 + \int_0^x c_2 dy - \int_0^x \int_0^y f(z) dz dy$$

$$= c_1 + c_2 x - \int_0^x \int_0^y f(z) dz dy \quad — ③$$

We will further simplify the double integral I ,

$$\text{Let } F(y) = \int_0^y f(z) dz \implies f'(y) = f(y)$$

$$\text{Then: } I = \int_0^x f(y) dy = \int_0^x y' F(y) dy$$

$$= y F(y) \Big|_0^x - \int_0^x y F'(y) dy$$

$$= x F(x) - \int_0^x y f(y) dy$$

$$\Downarrow I = x \int_0^x f(y) dy - \int_0^x y f(y) dy = \int_0^x (x-y) f(y) dy$$

Hence, the formula ③ is simplified as;

$$U(x) = C_1 + C_2 x - \int_0^x (x-y) f(y) dy$$

The constants C_1 and C_2 in the above formula can be determined from the boundary conditions ② as

$$0 = U(0) = C_1 \Rightarrow C_1 = 0$$

$$0 = U(1) = C_2 - \int_0^1 (1-y) f(y) dy \Rightarrow C_2 = \int_0^1 (1-y) f(y) dy$$

\therefore The solution of the 2-point BVP ② is given by,

$$U(x) = \int_0^1 x(1-y) f(y) dy - \int_0^x (x-y) f(y) dy \quad \text{--- (5)}$$

Define, $G(x,y) = \begin{cases} y(1-x), & \text{if } 0 \leq y \leq x \\ x(1-y), & \text{if } x \leq y \leq 1 \end{cases}$ --- (6)

Then check that the explicit formula ⑤ can be further compressed as

$$U(x) = \int_0^1 G(x,y) f(y) dy \quad \text{--- (7)}$$

Here, the function G is termed as a Green's function.

Ex. Check the following elementary properties of G .

① G is continuous.

② G is symmetric i.e. $G(x,y) = G(y,x)$

③ $G \geq 0$, $\forall x, y \in (0,1)$

④ For a fixed y , G is piecewise linear in x and vice versa

Thus, given the source term f , we can explicitly compute
 the solution u of the 2-point BVP (2) by integrating the
 formula (7), using the Green's function !!! (17)

However, even without explicitly ~~using~~ integrating, we can still obtain
 some interesting properties of the solution u as we see below.

1b. Smoothness of the Solution

In order to discuss the smoothness of solutions of the 2-point
 BVP (2), we need the following definition,

Defn: $C_0^k([0,1]) = \{g \in C^k([0,1]) : g(0) = g'$

Defn: $C([0,1]) = \{g : g \text{ is continuous on the closed
 interval } [0,1]\}$

$C^1([0,1]) = \{g : g \text{ is continuous on } (0,1); g' \text{ exists and is
 continuous on } (0,1)\}$

Denote the k th derivative of a function g as $g^{(k)}$.

Denote $g^{(1)} = g'$, $g^{(2)} = g''$ and so on.

$C_0^k([0,1]) = \{g : g \in C^k([0,1]) \cap C([0,1]) \text{ such that
 } g(0) = g'(0) = 0\}$.

We will use these spaces of smooth functions to
 measure the smoothness of the solution of (2). The precise
 statement follows.

(18)

Lemma: Let u be the soln of the 2-point BVP ② and ②'.

Then if $f \in C((0,1))$, $u \in C_0^2((0,1))$ and if

$f \in C^k((0,1))$, $u \in C_0^{k+2}((0,1))$.

Proof: u has the Green's function representation ⑦.

If f is continuous and G is clearly continuous, the integral implies that u is continuous. It is easy to check from ⑦ that

$$u(0) = u(1) = 0.$$

Also, differentiating inside the integral in ⑦, we obtain

$$\begin{aligned} u'(x) &= \int_0^1 \frac{\partial u}{\partial x}(x,y) f(y) dy \\ &= \int_0^1 (1-y)f(y) dy - \int_0^x f(y) dy \end{aligned}$$

Again, it is trivial to check that the above integral is continuous.

Furthermore differentiating once more;

$$u'' = -f$$

Thus f being continuous implies that u'' is continuous.

Hence, u, u', u'' are continuous and $u(0) = u(1) = 0$

thus, $u \in C_0^2((0,1))$.

Differentiating the BVP ② k -times:

$$-u^{(k+1)} = f^{(k)}.$$

Iterating the above argument, we obtain that

$$f \in C^k((0,1)) \Rightarrow u \in C^{k+2}((0,1)). \quad \square$$

(19)

1C. Maximum principles:

Another use of the representation formula ⑦ is to obtain a maximum principle. Again, we need a definition:

Defn: Given a continuous function g on $[0,1]$ ie $\cancel{g \in C}$
ie $g \in C([0,1])$.

Define

$$\|g\|_{\infty} = \max_{x \in [0,1]} |g(x)|.$$

The maximum principle for solutions of ② is given by.

Lemma: Assume that $f \in C([0,1])$ and u solves ②, then

$$\|u\|_{\infty} \leq \frac{1}{8} \|f\|_{\infty} \quad \text{--- ⑧}$$

Proof: By the formula ⑦

$$\begin{aligned} u(x) &= \int_0^1 G(x,y) f(y) dy \\ |u(x)| &= \left| \int_0^1 G(x,y) f(y) dy \right| \\ &\leq \int_0^1 |G(x,y)| |f(y)| dy \quad \left(\begin{array}{l} |\int f| \leq \int |f| \\ |ab| \leq |a||b| \end{array} \right) \\ &\leq \|f\|_{\infty} \int_0^1 G(x,y) dy \quad \left(\begin{array}{l} \text{Definition of norm} \\ G(x,y) > 0 \end{array} \right) \end{aligned}$$

By the definition of G :

$$G(x) \int_0^1 u(x,y) dy = \int_0^x y(1-x) dy + \int_x^1 x(1-y) dy.$$

(20)

$$= \frac{x^2}{2} \frac{(1-x)}{2} +$$

$$= \frac{x(1-x)}{2}$$

$$\therefore |U(x)| \leq \frac{\|f\|_{\infty} x(1-x)}{2}$$

$$\Rightarrow |U(x)| \leq \|f\|_{\infty} \frac{x(1-x)}{2} \quad \forall x \in (0,1) \\ \leq \frac{1}{8} \|f\|_{\infty} \quad (\text{as } \frac{x(1-x)}{2} \leq \frac{1}{8}, \forall x \in (0,1))$$

Taking a maximum over the LHS of the above

$$\max_{x \in [0,1]} |U(x)| \leq \frac{1}{8} \|f\|_{\infty}$$

$$\Rightarrow \|U\|_{\infty} \leq \frac{1}{8} \|f\|_{\infty} \quad \square$$

Remark: The estimate (8) tells us how large the solution can be if we know the "largeness" of the source f . We do not need to know the source exactly. Such "a-priori" estimates play a huge role in the analysis and numerical analysis of PDE as we will encounter later.

1d. Limitations of the Green's function representation

Although very useful, the Green's function representation is rather limited in scope due to the following factors.

- ⑥ The integral in the formula (7) is impossible to evaluate exactly for complicated source terms f .

(b) Green's function representation is only available for the Poisson's equation. In practice, we need to deal with generalized poisson's equations such as

$$-(a(x)u')' + b(x)u' + c(x)u = f \text{ on } (0,1)$$

$$u(0) = u(1) = 0 \quad \rightarrow \textcircled{9}$$

It is not possible to obtain Green's function representation for such problems $\textcircled{9}$ even for the very simple case of $b(x) \equiv c(x) \equiv 0$.

So, given the above limitations, we will use numerical schemes to approximate the solutions of the 2-point BVP $\textcircled{2}$ as well as the more general BVP $\textcircled{9}$.

2. Finite difference schemes for the 1-D poisson's equation

The PDE $\textcircled{2}$ on the domain $(0,1)$ is a "continuous" object. A computer can only be used to represent discrete objects. Therefore to discretize solve a PDE on a computer, we need to discretize the domain as well as the PDE itself.

2a: Discretization of the Domain:

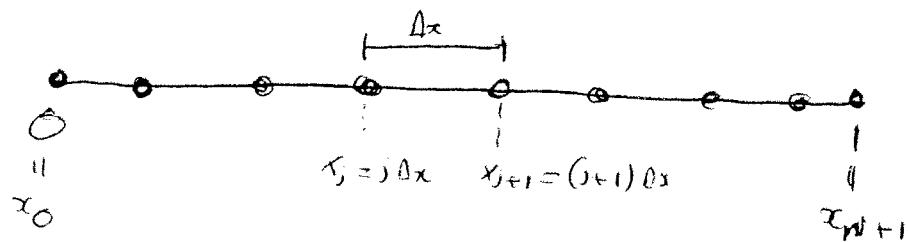
Given the set $[0,1]$, the simplest discretization is to divide it into equally spaced sub-intervals. To aid, we consider a fixed $\Delta x > 0$ and let

$$x_k = k\Delta x \text{ with } x_0 = 0$$

$$x_{(N+1)} = 1$$

And $\Delta x = \frac{1}{N+1}$, termed as the "mesh" size.

Thus the domain $[0, 1]$ is represented by a set of points, (22)
see figure 1.



2b. Discretizing the Derivatives

The 2-point BVP (2) involves "continuous" functions u and f .
The basic principle of a finite difference scheme is to represent
them as point-values i.e. define:

$$u_j \simeq u(x_j)$$

$$f_j = f(x_j)$$

The 2-point BVP also involves the derivatives of u . In a
finite difference (FD) scheme, these derivatives are replaced by
finite differences i.e.

$$u''(x_j) \simeq \frac{u(x_{j+1}) - 2u(x_j) + u(x_{j-1})}{\Delta x^2} \quad (10)$$

In fact the "errors" in this approximation can be readily
estimated.

Lemma: Let $u \in C^4([0, 1])$, then the error in the approximation
(10) can be estimated as:

(23)

$$\sup_{x \in (0,1)} \left| u''(x) - \frac{u(x+\Delta x) - 2u(x) + u(x-\Delta x)}{\Delta x^2} \right| \leq \frac{\|u^{(4)}\|_\infty}{12} \Delta x^2 \quad \text{--- (10)}$$

Proof: As $u \in C^4(0,1)$, expand u in a Taylor's series around x to obtain the desired estimate. ~~with~~

2c. Definition of the Scheme.

Combining the above ingredients, we can formulate the following finite difference scheme:

$$\forall j = 2, \dots, N-1;$$

$$-u_{j-1} + 2u_j - u_{j+1} = \Delta x^2 f_j$$

using the boundary condition at 0;

$$2u_1 - u_2 = \Delta x^2 f_1, \quad u_0 = 0$$

using the boundary condition at 1;

$$-u_{N-1} + 2u_N = \Delta x^2 f_{N-1}, \quad u_{N+1} = 0$$

The above set of equations is a linear equal system of N unknowns (u_1, u_2, \dots, u_N) , augmented with N equations.

∴ we can represent it as a Matrix equations:

$$AU = F \quad \text{--- (11)}$$

Here: $U = (u_1, u_2, \dots, u_N)$ is a N -vector.

$F = \Delta x^2 (f_1, \dots, f_N)$ is a N -vector.

$$A = \begin{pmatrix} 2 & -1 & 0 & \dots & 0 \\ -1 & 2 & -1 & \dots & 0 \\ 0 & -1 & 2 & -1 & \dots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 2 \end{pmatrix}$$

Remark: Thus, approximating a differential equation (2) reduces to solving a "very large" linear system (11).

2d. Solution of the linear system (11)

The matrix A in the linear system (11) is a tridiagonal, diagonally dominant matrix. Furthermore, it is symmetric, positive-definite and hence invertible. Any direct numerical linear algebra solver such as LU decomposition can be used.

2e. Numerical results

See accompanying slides.

Conclusions

1. The discrete (computed) solutions are stable.
2. The computed solutions converge to the exact solution of (2) as $\Delta x \rightarrow 0$
3. The experimental order of convergence is 2.

3. Convergence analysis of the finite difference scheme.

We will seek to explain the numerical results by an analysis of the finite difference Scheme (11).

For simplicity of notation, we introduce the following terminology:

$$D_0^{\Delta x} = \{(\omega_0, \omega_1, \dots, \omega_N, \omega_{N+1}) \text{ with}$$

$$\omega_0 = \omega_{N+1} = 0$$

$$\omega_j = \omega(j\Delta x) \text{ and } \omega$$

is a continuous function on $(0, 1)$.

$$L^{\Delta x}: D^{\Delta x} \mapsto D^{\Delta x}: \omega^{\Delta x} = (\omega_1, \dots, \omega_N)$$

such that

$$L^{\Delta x} \omega_j = -\frac{(\omega_{j+1} - 2\omega_j + \omega_{j-1})}{\Delta x^2}$$

i.e. the finite difference scheme can be succinctly represented as:

$$L^{\Delta x} u^{\Delta x} = f^{\Delta x} \quad \text{--- (12)}$$

$$u_0 = u_{N+1} = 0$$

Thus the Operator $L^{\Delta x}$ is an approximation of the Laplacian $(-\Delta)$.

3a. Stability of $L^{\Delta x}$:

We know that the continuous solution operator was stable. let's recall:

(26)

$$-\Delta u = f$$

$$\text{Then } u = (-\Delta)^{-1} f$$

From the maximum principle (8),

$$\|u\|_{\infty} \leq \frac{1}{8} \|f\|_{\infty}$$

$$\therefore \|(-\Delta)^{-1} f\|_{\infty} \leq \frac{1}{8} \|f\|_{\infty}$$

$$\text{Formally: } \|(-\Delta)^{-1} f\|_{\infty} \approx \|(-\Delta)^{-1}\| \|f\|_{\infty}$$

Here $\|(-\Delta)^{-1}\|$ is an operator norm.

\therefore The ~~discrete~~ maximum principle provides some stability to $(-\Delta)^{-1}$.

We will attempt to prove a similar stability estimate for the discrete operator $L^{\Delta x}$.

To this end, fix a grid point x_k and define the discrete Green's functions G^k as:

$G^k(x_j) = G(x_j, x_k)$, Here G is the Green's function defined earlier.

As $G(x, y)$ is linear in x for $x \neq y$, it is easy to check that

$$L^{\Delta x} G^k(x_j) = 0, \quad \forall j \neq k.$$

$$L^{\Delta x} G^k(x_k) = -\frac{1}{\Delta x^2} (G^k(x_k + \Delta x) - 2G^k(x_k) + G^k(x_k - \Delta x))$$

$$= -\frac{1}{\Delta x^2} (G(x_{k+\Delta x}, x_k) - 2G(x_k, x_k) + G(x_{k-\Delta x}, x_k)).$$

$$\therefore = -\frac{1}{\Delta x^2} \left(x_k (1-x_k - \Delta x) - 2x_{k-1} (1-x_k) + (x_{k-1} - \Delta x) (1-x_{k-1}) \right)$$

$$= \frac{1}{\Delta x},$$

(27)

∴ we can rewrite:

$$L^{\Delta x} G^k = \frac{1}{\Delta x} e^k, \text{ with}$$

$$e^k = \begin{cases} 1 & \text{if } k \\ 0 & \text{if } k \neq k \end{cases}$$

$$e^k(x_j) = \begin{cases} 1 & \text{if } j=k \\ 0 & \text{if } j \neq k \end{cases}$$

Discrete Green's function representation:

↳ for any $f \in P_0^{\Delta x}$, define ω as:

$$\omega = \Delta x \sum_{k=1}^N f_k G^k$$

$$L^h \omega = L^h \left(\Delta x \sum_{k=1}^N f_k G^k \right).$$

Applying the operator $L^{\Delta x}$ to the above expression;

$$L^{\Delta x} \omega = L^{\Delta x} \left(\Delta x \sum_{k=1}^N f_k G^k \right).$$

$$= \Delta x \sum_{k=1}^N f_k L^{\Delta x}(G^k) \quad (\text{by linearity}) \quad \underline{\text{and}}$$

$$= \Delta x \cdot \sum_{k=1}^N f_k e_k = f$$

∴ The solution of the difference scheme (12) can be written as a discrete green's function.

$$u^{\Delta x} = u^{\Delta x}(x_j) = \Delta x \sum_{k=1}^N G(x_j, x_k) f(x_k) \quad \text{--- (13)}$$

We will now use the discrete Green's function representation (28) to prove the discrete maximum principle:

Let $f \in \mathbb{D}^h$, then define:

$$\|f\|_{\Delta x, \infty} = \sup_{1 \leq j \leq N} |f_j| = \sup_{1 \leq j \leq N} |f(x_j)|.$$

Lemma (Discrete Maximum principle): Let $u^{\Delta x}$ be solution of the finite difference scheme (12), then it satisfies:

$$\|u^{\Delta x}\|_{\Delta x, \infty} \leq \frac{1}{8} \|f^{\Delta x}\|_{\Delta x, \infty} \quad - (14)$$

Proof: By the Green's function representation:

$$u_j = u(x_j) = \Delta x \sum_{k=1}^N f_k g(x_j, x_k)$$

$$\text{Hence } |u_j| \leq \Delta x \|f^{\Delta x}\|_{\Delta x, \infty} \sum_{k=1}^N g(x_j, x_k)$$

It is easy to check that: $\Delta x \sum_{k=1}^N g(x_j, x_k) = \frac{x_j(1-x_j)}{2}$

$$\begin{aligned} |u_j| &\leq \frac{1}{8} \|f^{\Delta x}\|_{\Delta x, \infty} \frac{x_j(1-x_j)}{2} \\ &\leq \frac{1}{8} \|f^{\Delta x}\|_{\Delta x, \infty} \quad (\text{as } \max_j \frac{x_j(1-x_j)}{2} = \frac{1}{8}) \end{aligned}$$

Hence;

$$\|u^{\Delta x}\|_{\Delta x, \infty} \leq \frac{1}{8} \|f^{\Delta x}\|_{\Delta x, \infty}$$

Hence, a small change in the source will lead to a corresponding change in the solution indicating that the solution is stable.

(29)

3.2 Consistency of the discrete solution:

To quantify the error in the solution of the Scheme (12), we need to define:

Truncation error: Let $f^{\Delta x}$ be a discrete representation of source f i.e. $f^{\Delta x} = \{f_j\}_{j=1}^N$, $f_j = f(x_j)$

Similar let U be a discrete representation of the soln u of the 2-point BVP (2):

$$U = \{u_j\}_{j=1}^N \quad u_j = u(x_j).$$

Then the truncation error $C^{\Delta x} = \{C_j\}_{j=1}^N$ is defined as

~~$$C_j = L^{\Delta x} U - f^{\Delta x}$$~~ — (16)

Lemma: ~~The fact~~ If $f \in C^2(0,1)$, then the truncation error $C^{\Delta x}$ can be estimated as:

$$\|C^{\Delta x}\|_{\Delta x, \infty} \leq \frac{\|f''\|_{\infty}}{12} \Delta x^2 \quad (15)$$

Proof: As $f \in C^2(0,1)$, by the smoothness of solutions $u \in C_0^4(0,1)$. Furthermore:

$$\|u'''\|_{\infty} = \|f''\|_{\infty}$$

by the definition of truncation error, this

$$\begin{aligned} C^{\Delta x}(x_j) &= L^{\Delta x} u(x_j) - f(x_j) \\ &= -\frac{1}{\Delta x^2} \underbrace{(u(x_{j+1}) - 2u(x_j) + u(x_{j-1}))}_T - f(x_j) \end{aligned}$$

(30)

As $u \in C_0^4(0,1)$, performing a Taylor expansion of u at x_j and identifying $u''(x_j) = f(x_j)$
 $u'''(x_j) = f'''(x_j)$, the term T is:

$$|T| = \left| -u''(x_j) - f(x_j) + \frac{u'''(\theta)}{12} \Delta x^3 \right| \quad \text{with:} \\ \theta \in [x_{j-1}, x_{j+1}]$$

$$\Rightarrow |T| = \frac{\Delta x^4}{12} |u'''(\theta)| \leq \frac{\Delta x^4}{12} \|f''\|_\infty$$

$$\therefore |C^\Delta x(x_j)| \leq \frac{\Delta x^2}{2} \|f''\|_\infty \quad \square.$$

Furthermore, estimate (5) now leads to:

$$\lim_{\Delta x \rightarrow 0} C^\Delta x = 0$$

In other words, as the mesh is refined, the truncation error vanishes and the scheme is consistent.

3c. Convergence: we can combine the stability estimate (14) and the consistency estimate (15) to obtain ~~the~~ convergence of the approximate solution $u^\Delta x$ of the scheme (12) to the exact solution u of (2).

Define the errors:

~~$u^\Delta x = f_{\Delta x}$~~ $E^\Delta x = \{E_j\}_{j=1}^N$ with

$$E_j = u(x_j) - u^\Delta x(x_j)$$

$$\therefore E^\Delta x = U - u^\Delta x$$

Applying the scheme operator $L^{\Delta x}$ to $E^{\Delta x}$, (31)

$$\begin{aligned}
 L^{\Delta x}(E^{\Delta x}) &:= L^{\Delta x}(U - u^{\Delta x}) \\
 &= L^{\Delta x}U - L^{\Delta x}u^{\Delta x} \quad (\text{linearity of } L^{\Delta x}) \\
 &= L^{\Delta x}U - f^{\Delta x} \quad (\text{by defn of scheme (12)}) \\
 &= C^{\Delta x} \quad (\text{by defn of truncation error (10)}) \\
 \therefore L^{\Delta x}E^{\Delta x} &= C^{\Delta x} \quad - (17)
 \end{aligned}$$

Note that (17) is exactly the same as scheme (12) with $C^{\Delta x}$ playing the role of sd source $f^{\Delta x}$ and the error $E^{\Delta x}$ replacing the discrete solution $U^{\Delta x}$.

Now recalling the discrete maximum principle (14), we obtain

$$\begin{aligned}
 \|E^{\Delta x}\|_{\Delta x, \infty} &\leq \frac{1}{8} \|C^{\Delta x}\|_{\Delta x, \infty} \\
 &\leq \frac{\Delta x^2}{96} \|f^{\Delta x}\|_{\infty} \quad (\text{by the consistency estimate (15)})
 \end{aligned}$$

$\therefore E^{\Delta x} \rightarrow 0$ as $\Delta x \rightarrow 0$

or: $u^{\Delta x} \rightarrow u$ (exact solution) as $\Delta x \rightarrow 0$.

Furthermore the rate of convergence is 2. This justifies the observations of numerical experiments.

Thus: Stability + Consistency \implies Convergence.

A more general version of the above statement is known as the Lax Equivalence theorem.