

Poisson's equation in 2-D

Let $\Omega \subseteq \mathbb{R}^2$ be a domain. We are interested in solutions of the Poisson equation:

$$\begin{aligned} -\Delta u &= f \text{ on } \Omega & \text{--- (1)} \\ u &\equiv 0 \text{ on } \partial\Omega \text{ (boundary of } \Omega) \end{aligned}$$

Explicit solution formulas such as Green's function representation are only available for very specific domains, say for instance if Ω is the unit circle: $\Omega = B(0,1) = \{ (x,y) \in \mathbb{R}^2 \text{ with } x^2 + y^2 < 1 \}$.

Similarly, we can only formulate finite difference schemes for the simple domain like the unit square: $[0,1]^2$: (figure 1)

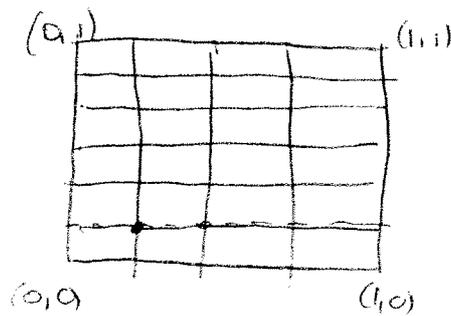


figure 1

Fix $\Delta x, \Delta y > 0$, discretize the interval $[0,1]$ (x-axis) into $(N+2)$ points such that: $x_0 = 0, x_{N+1} = 1, x_i = i \Delta x, \forall 1 \leq i \leq N, \Delta x = \frac{1}{N+1}$

Similarly discretize $[0,1]$ (y-axis) into $(M+2)$ points such that $y_0 = 0, y_{M+1} = 1, y_j = j \Delta y, \forall 1 \leq j \leq M, \Delta y = \frac{1}{M+1}$.

We are interested in approximating the point values of the solution u of (1) i.e.

$$u_{ij} \approx u(x_i, y_j)$$

To do this we define $f_{ij} = f(x_i, y_j)$ and we need to discretize the Laplace operator:

$$-\Delta u = -(u_{xx} + u_{yy})$$

(4)

By using a central difference approximation,

$$u_{xx}(x) \approx \frac{u(x) - 2u(x) + u(x)}{\Delta x^2}$$

$$u_{xx}(x_i, y_j) \approx \frac{u(x_i + \Delta x, y_j) - 2u(x_i, y_j) + u(x_i - \Delta x, y_j)}{\Delta x^2}$$

$$\approx \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{\Delta x^2}$$

Similarly: $u_{yy}(x_i, y_j) \approx \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{\Delta y^2}$

Hence, the finite difference approximation of (1) on a unit square is:

$$-\left(\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{\Delta x^2} + \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{\Delta y^2} \right) = f_{i,j} \quad (2)$$

$$\forall 1 \leq i, j \leq N.$$

Similarly the boundary conditions are:

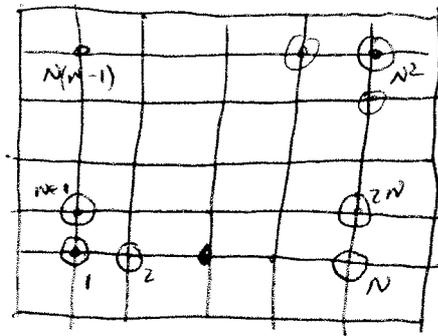
$$u_{0,j} = u_{N+1,j} \equiv 0, \quad \forall j = 0, 1, \dots, M+1$$

$$u_{i,0} = u_{i,M+1} \equiv 0, \quad \forall i = 0, 1, \dots, N+1$$

In the special case of $\Delta x = \Delta y$, the scheme can be realized by considering the vectors:

$$U = \{u_1, u_2, \dots, u_{N+1}, u_2, \dots, u_{N+1}, \dots, u_N, \dots, u_{NN}\}$$

see the numbering in figure (2)



and feed vector:

$$F = \Delta x^2 \{ f_{11}, \dots, f_{N1}, \dots, f_{1N}, \dots, f_{NN} \}.$$

and matrix

$$A = \begin{pmatrix} 4 & -1 & 0 & \dots & 0 & -1 & 0 & \dots & 0 \\ -1 & 4 & -1 & & & 0 & -1 & & \\ 0 & -1 & 4 & -1 & & & 0 & -1 & \dots & 0 \\ & & & & & & & & & \\ 0 & 0 & \dots & & -1 & 0 & & & -1 & 4 & -1 \\ & & & & 0 & -1 & 0 & \dots & & -1 & 4 \end{pmatrix}$$

as the matrix equation $AU = F$:

The matrix is non-singular and sparse and can be inverted by standard numerical linear algebra routines.

Numerical experiments and convergence analysis of the Scheme (2) can be performed, in analogy with the one-dimensional case.

However, finite differences are limited in 2 dimensions due to

1. Very difficult to provide a finite difference mesh for general domains.
2. It is very difficult to impose suitable boundary conditions.

The above proof of convergence of the numerical scheme (12) to (32) the exact solution of the Poisson's equation uses the discrete Green's function representation heavily. One can say that this is a some sort of cheating as we have argued before that the Green's function representation is only available for the particular case of the Poisson's equation but not for more general equations like:

$$-(a(x)u')' + b(x)u' + c(x)u = f, \quad x \in (0,1). \\ u(0) = 0 = u(1) \quad \text{--- (18)}$$

However, as argued before numerical schemes can readily be used for the above equation. How does one analyse the numerical scheme in the absence of explicit formulas like the Green's function representation

• The Energy method

Although the energy method is applicable to more general elliptic 2-point BVPs like (18), we will consider it in the context of the Poisson's equation:

$$-u'' = f \quad \text{on } (0,1) \quad \text{--- (2)} \\ u(0) = u(1) = 0$$

lets multiply ~~both~~ both sides of (2) by u :

$$-u''u = uf$$

Integrate the above on the interval $(0,1)$

$$-\int_0^1 u(x) u''(x) dx = \int_0^1 u(x) f(x) dx.$$

By integrating by parts the lhs of above:

(33)

$$\int_0^1 |u'(x)|^2 dx - u(1)u'(1) + u(0)u'(0) = \int_0^1 u(x)f(x) dx.$$
$$\Rightarrow \int_0^1 |u'|^2 dx = \int_0^1 uf dx \quad - (19)$$

Note that in the above, we have used the boundary condition $u(0) = u(1) = 0$ and suppressed x in the integral for notational convenience.

using the elementary inequality: $ab \leq \frac{a^2}{2} + \frac{b^2}{2}$, we can write (19) as:

$$\int_0^1 |u'|^2 dx \leq \frac{1}{2} \int_0^1 u^2 dx + \frac{1}{2} \int_0^1 f^2 dx. \quad - (20)$$

Thus to get further information from (20), we will use the

following trick:

Consider any $u \in C_0^1([0,1])$:

Then, $u(x) = u(0) + \int_0^x u'(s) ds$ (Fundamental theorem of integral calculus)

$$\Rightarrow u(x) = \int_0^x u'(s) ds \quad (\text{as } u(0) = 0)$$

$$\Rightarrow |u(x)| \leq \int_0^x |u'(s)| ds \leq \int_0^1 |u'(s)| ds.$$

$$\Rightarrow |u(x)| \leq \int_0^1 1 \cdot |u'(s)| ds \quad \text{--- (21)}$$

Now we will use a famous integral inequality known as the Cauchy-Schwarz inequality:

$$\int_0^1 g(s)h(s) ds \leq \left(\int_0^1 g^2(s) ds \right)^{1/2} \left(\int_0^1 h^2(s) ds \right)^{1/2}$$

Using Cauchy-Schwarz in (21) with $g=1$ and $h=|u'|$, we obtain:

$$|u(x)| \leq \left(\int_0^1 |u'(s)|^2 ds \right)^{1/2} \quad \text{--- (22)}$$

$$\Rightarrow |u(x)|^2 \leq \int_0^1 |u'(s)|^2 ds.$$

Now integrating both sides of the above over $(0,1)$ and noting that the rhs is independent of x , we obtain:

$$\int_0^1 |u(x)|^2 dx \leq \int_0^1 |u'(s)|^2 ds$$

$$\Rightarrow \int_0^1 |u(x)|^2 dx \leq \int_0^1 |u'(x)|^2 dx \quad \text{--- (23)}$$

Note that (23) controls the a measure of the integral of the function in terms of the derivative and is termed as a Poincare inequality.

by applying (23) to the estimate (26):

$$\int_0^1 |u'|^2 dx \leq \frac{1}{2} \int_0^1 |u''|^2 dx + \frac{1}{2} \int_0^1 f^2 dx$$

$$\Rightarrow \int_0^1 |u'|^2 dx \leq \int_0^1 f^2 dx \quad (29)$$

In view of (23) and of (22), we further obtain:

$$\int_0^1 u^2 dx \leq \int_0^1 f^2 dx$$

$$\int_0^1 |u'|^2 dx \leq \int_0^1 f^2 dx$$

and

$$|u(x)| \leq \left(\int_0^1 f^2(x) dx \right)^{1/2}, \quad \forall x \in (0,1).$$

The above inequalities can be expressed in terms of certain norms. To this end, we define:

~~$$L^2(0,1) = \left\{ g : \int_0^1 g^2(x) dx \leq C \right\}$$~~

$$\|u\|_{L^2} = \left(\int_0^1 |u(x)|^2 dx \right)^{1/2}$$

$$\|u\|_{H_0^1} = \left(\int_0^1 |u'(x)|^2 dx \right)^{1/2}$$

$$\|u\|_{L^\infty} = \sup_{x \in [0,1]} |u(x)|.$$

and the function spaces:

$$L^2(0,1) = \{ \text{All functions } g : \|g\|_2 \leq C \}$$

$$H_0^1(0,1) = \{ \text{All functions } g \text{ such that } g(0) = 0 = g(1) \text{ and } \|g\|_{H_0^1} \text{ is finite} \}$$

$$L^\infty(0,1) = \{ \text{All functions } g : \|g\|_{L^\infty} \text{ is finite} \}$$

These function spaces will play a key role in the study of PDEs and of numerical methods for them.

We can further summarize the estimates (22) to (24) as:

$$\begin{aligned} \|u\|_2 &\leq \|f\|_2 \\ \|u\|_{H_0^1} &\leq \|f\|_2 \quad - (25) \\ \|u\|_{L^\infty} &\leq \|f\|_2 \end{aligned}$$

Furthermore the estimate (22) and the Poincaré inequality (23) can be written as:

$$\begin{aligned} \|u\|_{L^\infty} &\leq \|u\|_{H_0^1} \\ \|u\|_2 &\leq \|u\|_{H_0^1} \end{aligned}$$

⊙ Uniqueness: An immediate consequence of the energy estimates (25) is the following Lemma:

Lemma (Uniqueness): The solutions of the Poisson's equation (2) are unique.

Proof: Let u, v be two solutions of the Poisson's equation:

Let then the difference w satisfies:

$$\begin{aligned} -w'' &= 0 \\ w(0) &= w(1) = 0 \end{aligned}$$

i.e. Poisson's equation with rhs 0. So we can apply the energy estimate (25) to obtain

Variational formulation:

(96)

Considers the one-dimensional Poisson equation:

$$\begin{aligned} -u'' &= f \quad \text{on } (0,1) \\ u(0) &= u(1) = 0. \end{aligned} \quad \text{--- (1)}$$

We call (1) as the strong form of the PDE.

From the discussion on the energy method, we know that:

$$u \in H_0^1(0,1) \text{ if } f \in L^2(0,1)$$

furthermore by Poincaré inequality; $u \in L^2(0,1)$; Therefore we can define the following functional;

$$J(w) := \frac{1}{2} \int_0^1 |w'|^2 dx - \int_0^1 wf dx, \quad \forall w \in H$$

Note that: $J(w)$ is well defined as.

$$\begin{aligned} |J(w)| &= \left| \frac{1}{2} \int_0^1 |w'|^2 dx - \int_0^1 wf dx \right| \\ &\leq \frac{1}{2} \|w'\|_{H_0^1}^2 + \left| \int_0^1 wf dx \right| \leq \frac{1}{2} \|w'\|_{H_0^1}^2 + \frac{1}{2} \|w'\|_{L^2} \|f\|_{L^2} \\ &\quad \text{(Cauchy-Schwarz)} \\ &\leq \frac{1}{2} \|w'\|_{H_0^1}^2 + \|w'\|_{H_0^1} \|f\|_{L^2} \end{aligned}$$

Let:

~~u~~ u be the minimizer of the

Variational problem for J i.e.

$$u \in H_0^1(0,1) : J(u) \leq J(w), \quad \forall w \in H_0^1(0,1)$$

(Poincaré inequality)

formally:

(7)

$$J'(u) = \lim_{\epsilon \rightarrow 0} \frac{J(u + \epsilon v) - J(u)}{\epsilon} = 0$$

$$\forall v \in H_0^1(0,1)$$

Performing this computation as in chapter (1), we obtain:

$$\int_0^1 u'(x) v'(x) dx = \int_0^1 v(x) f(x) dx. \quad \text{--- (3)}$$
$$\forall v \in H_0^1(0,1)$$

The above form (3) is termed as the weak or variational form of (1), & this equality is called principle of virtual work in physics.

Note that (3) is well defined for $u \in H_0^1$ as the lhs is well-defined by Cauchy-Schwarz inequality i.e.

$$\int_0^1 u' v' dx \leq \|u\|_{H_0^1} \|v\|_{H_0^1}$$

Similarly:

$$\int_0^1 v f dx \leq \|v\|_{L^2} \|f\|_{L^2} \leq \|v\|_{H_0^1} \|f\|_{L^2} \quad (\text{Poincaré}).$$

However, if u has further regularity, we can use integration by parts on (3) and obtain.

$$-\int_0^1 (u'' + f)v dx = 0 \quad \forall v \in H_0^1$$

$\Rightarrow -u'' = f$ and retrieve the strong form of the Poisson problem.

Recall that we derived the Poisson Equation from (98) minimizing the functional (Dirichlet energy). Thus, the form (3) is more fundamental than the strong form (1). In fact, we will discretize the form (3) in a suitable manner.

Note that the ^{energy} estimate can be easily obtained from (3) by substituting: $v = u$ in (3) i.e.

$$\begin{aligned} \int |u'|^2 dx &= \int u f dx \\ &\leq \frac{1}{2} \int u^2 dx + \frac{1}{2} \int f^2 dx \\ &\leq \frac{1}{2} \int |u'|^2 dx + \frac{1}{2} \int f^2 dx \quad (\text{Poincaré}) \\ \Rightarrow \int |u'|^2 dx &\leq \int f^2 dx \quad (\text{stability}) \\ \text{and } \int u^2 dx &\leq \int f^2 dx \end{aligned}$$

Finite Element formulation: Let us recall the variational form of the Poisson's problem. Let $V = H_0^1(0,1) = \{ \forall v \text{ continuous such that } \int_0^1 |v'|^2 dx \leq C \text{ and } v(0) = v(1) = 0 \}$, then we seek a

(4) $u \in V$ such that $\forall v \in V$,

$$\int u' v' dx = \int f v dx :$$

For simplicity of notation, we denote the "inner product"

$$(u, v) = \int_0^1 u(x) v(x) dx$$

Then, the variational formulation is

(99)

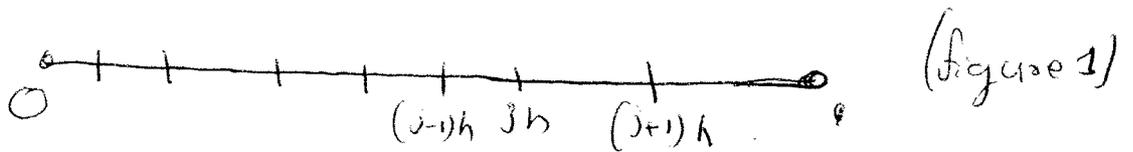
$$\textcircled{4} \text{ find } u \in V \text{ such that } \forall v \in V \\ (u', v') = (f, v)$$

In the finite element method (FEM), we will discretize the variational formulation of the Poisson problem $\textcircled{4}$ by

"Find a finite dimensional subspace $V_h \subset V$ "

In other words, we will replace V (infinite dimensional) by a finite dimensional subspace V_h .

To this end, discretize the interval $(0,1)$ exactly as before. For $h > 0$, do let: $h = \frac{1}{N+1}$ and define $x_0 = 0, x_{N+1} = 1$ and $x_j = jh$: (see figure 1)

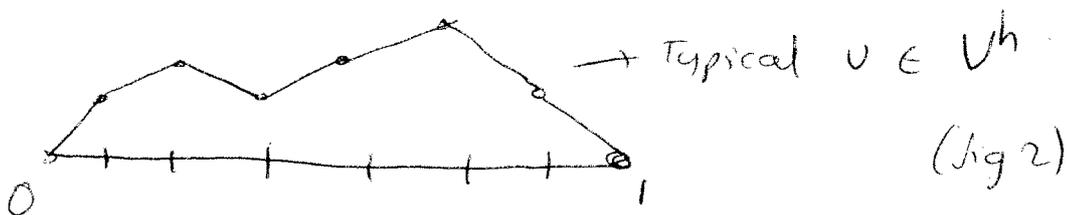


and let

$$V_h = \left\{ v \text{ is continuous on } (0,1) \text{ and } v|_{[(j-1)h, jh]} \text{ is linear} \right\}$$

$v_0 = v_1 = 0$

ie. V_h is all continuous piecewise linear functions on $(0,1)$



FEM consists of solving the following discrete

Variational problem:

⑤ find a $u_h \in V^h$ such that for all $v \in V^h$,

$$(u_h, v) = (f, v), \forall v \in V^h.$$

Concrete realization. To see if the discrete variational problem has a solution, we readily check that the functions:

$$\begin{aligned} \Phi_j(x_i) &= 1, \text{ if } i=j \\ &= 0, \text{ if } i \neq j \end{aligned} \quad \text{see figure 3}$$

Φ_j is piecewise linear, continuous, i.e. $\Phi_j \in V^h$

form a basis for V^h : i.e.

any $v \in V^h$ can be represented as:

$$v = \sum_{j=1}^N n_j \Phi_j(x), \text{ with } n_j = v(x_j)$$

∴ from ⑤ $\Leftrightarrow (u_h, v) = (f, v)$

$$\Leftrightarrow (u_h, (\sum n_j \Phi_j)) = (f, \sum n_j \Phi_j)$$

$$\Leftrightarrow \sum_{j=1}^N n_j (u_h, \Phi_j) = \sum_{j=1}^N n_j (f, \Phi_j)$$

⑥ $\Leftrightarrow (u_h, \Phi_j) = (f, \Phi_j) \forall 1 \leq j \leq N$

Furthermore, as ~~the~~ $u_h \in V_h$, we have that

$$u_h = \sum_{i=1}^N u_i \varphi_i$$

Hence from (6), we obtain: $\forall 1 \leq j \leq N$

$$\sum_{i=1}^N u_i (\varphi_i', \varphi_j') = (f, \varphi_j')$$

Here define the Stiffness matrix:

$$A = \{A_{ij}\}_{i,j=1,\dots,N} \text{ with}$$

$$A_{ij} = (\varphi_i', \varphi_j')$$

define the load vectors as:

$$F = \{f_j\}_{j=1}^N, \quad f_j = (f, \varphi_j')$$

and solution vectors as:

$$U = \{u_j\}_{j=1}^N$$

(6) reduces to a matrix equation:

$$AU = F \quad (7)$$

The matrix A is clearly symmetric.

Consider a vector $\omega \in \mathbb{R}^N$, then

$$\langle A\omega, \omega \rangle = \sum_{i,j=1}^N \omega_i A_{ij} \omega_j = \sum_{i,j=1}^N \omega_i (\varphi_i', \varphi_j') \omega_j$$

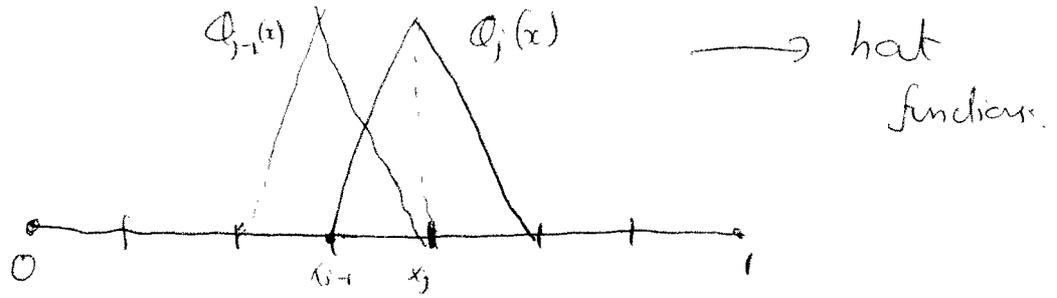
$$\Leftrightarrow = \left(\left(\sum_{i=1}^N \omega_i \varphi_i' \right)', \left(\sum_{j=1}^N \omega_j \varphi_j' \right)' \right)$$

$$\text{let } v = \sum_{i=1}^N \omega_i \varphi_i' \Rightarrow (v', v') = \int v'^2 dx > 0 \text{ if } (v \neq 0)$$

(5)

Hence A is a positive definite symmetric matrix and is invertible. Thus the discrete variational problem (6) has a Unique Solution.

Computing the stiffness matrix.



It is easy to see that

$$A_{ij} = (\phi_i', \phi_j') \equiv$$

$$\begin{aligned} \phi_j'(x) &= \begin{cases} \frac{1}{h} & \text{if } x \in [x_{j-1}, x_j] \\ -\frac{1}{h} & \text{if } x \in [x_j, x_{j+1}] \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

$$\therefore A_{ij} = (\phi_i', \phi_j') \equiv 0 \text{ if } |i-j| > 1$$

$$A_{j-1,j} = (\phi_{j-1}', \phi_j') = -\frac{1}{h} = (\phi_j', \phi_{j+1}') = A_{j,j+1}$$

$$\begin{aligned} A_{j,j} = (\phi_j', \phi_j') &= \int_{x_{j-1}}^{x_j} \frac{1}{h^2} dx + \int_{x_j}^{x_{j+1}} \frac{1}{h^2} dx \\ &= \frac{1}{h} + \frac{1}{h} = \frac{2}{h} \end{aligned}$$

$$\therefore A = \frac{1}{h} \begin{pmatrix} 2 & -1 & \dots & 0 & 0 \\ -1 & 2 & \dots & \dots & 0 \\ \dots & \dots & -1 & 2 & -1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & -1 & 2 \end{pmatrix}$$

In other words, upto a scaling the FEM stiffness matrix is the same as the matrix that arises in the finite difference method !!!

The load vector is slightly different from the finite difference method. However if we choose to evaluate the load vector as by the midpoint rule, i.e.

$$\begin{aligned}
 \mathbf{F} &= \{f_j\}_{j=1}^N & f_j &= \int_0^r f(x) \phi_j(x) dx \\
 & & &= \int_{x_{j-1}}^{x_{j+1}} f(x) \phi_j(x) dx \quad \text{as } \phi_j \neq 0 \text{ in } [x_{j-1}, x_{j+1}] \\
 & & &\approx f\left(\frac{x_{j-1}+x_{j+1}}{2}\right) \phi_j\left(\frac{x_{j+1}-x_{j-1}}{2}\right) (x_{j+1}-x_{j-1}) \\
 & & &\approx 2h f(x_j)
 \end{aligned}$$

~~The finite element~~

Convergence analysis

Let u be the exact solution of (4):

and u_h be the FEM solution (i.e. it solves (5) with fixed n)

Define error $e := u - u_h$, clearly $e \in V$.

Now by (4) $(u', v') = (f, v), \forall v \in V$

$\Rightarrow (u', v') = (f, v), \forall v \in v_h \subset V$

from (5) $(u_h', v') = (f, v) \quad \forall v \in v_h$

subtracting the above 2-expressions;

$$(u' - u_n', v') = (f, v) \quad \forall v \in V^h \quad (59)$$

$$(u - u_n)', v') = (f, v), \quad \forall v \in V^h \quad (\text{linearity of derivative})$$

$$\Rightarrow (e', v') = 0, \quad \forall v \in V^h \quad (\text{defn of } e)$$

The above relation is termed as Galerkin orthogonality i.e. the error is orthogonal to the subspace V^h .

Let for any $v \in V^h$; define $w := u^h - v \in V^h$.

$$\|e\|_{H_0^1}^2 = \int_0^1 |e'|^2 dx$$

$$= (e', e')$$

$$= (e', e') + (e', w') \quad (\text{Galerkin orthogonality})$$

$$= (e', (e+w)') \quad (\text{linearity})$$

$$= (e', (u - u^h + u^h - v)') \quad (\text{defn})$$

$$= (e', (u - v)')$$

$$= \int_0^1 e' (u - v)' dx$$

$$\leq \left(\int_0^1 |e'|^2 dx \right)^{1/2} \left(\int_0^1 |u - v|^2 dx \right)^{1/2} \quad (\text{Cauchy-Schwarz})$$

$$\leq \|e\|_{H_0^1} \|u - v\|_{H_0^1}$$

$$\Rightarrow \|e\|_{H_0^1} \leq \|u - v\|_{H_0^1} \quad \forall v \in V^h$$

$$\Rightarrow \|u - u_n\|_{H_0^1} \leq \|u - v\|_{H_0^1}, \quad \forall v \in V^h \quad \text{--- (7)}$$

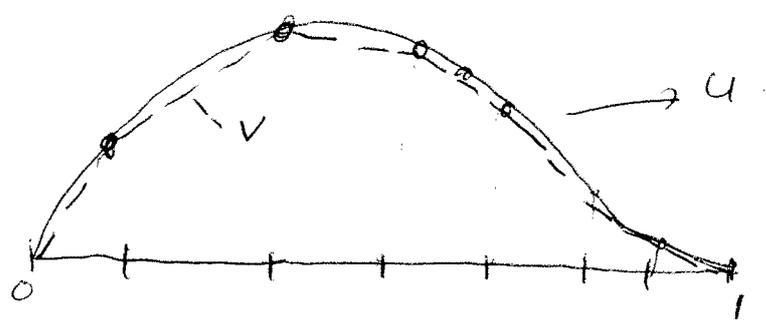
Also by

Thus the FEM error is bounded by the.

In particular, we can choose v to be the piecewise linear interpolant of u . i.e. on a given grid, (see figure 9), v is a piecewise linear continuous function such that:

$$v(0) = v(1) = 0 \text{ and}$$

$$v(x_j) = u(x_j).$$



Thus such an "Interpolant" is often denoted as $I_h u$:

from elementary numerical analysis, we know that

$$\|u'(x) - I_h(u'(x))\| \leq h \max_{0 \leq y \leq 1} |u''(y)| \quad \text{--- (7)}$$

$$|u(x) - I_h u(x)| \leq \frac{h^2}{8} \max_{0 \leq y \leq 1} |u''(y)|$$

clearly by squaring and integrating, we obtain:

$$\|u - I_h u\|_{H_0^1} \leq Ch \|u''\|_{L^2} \quad \text{--- (8)}$$

now combining estimates (8) and (7), we obtain:

$$\|u - u_h\|_{H_0^1} \leq Ch \|u''\|_{L^2}$$

from Poincaré inequality, we also obtain:

$$\|u - u_h\|_{L^2} \leq Ch \|u''\|_{L^2}.$$

$$\therefore \lim_{h \rightarrow 0} u^h = u$$

And the FEM converges!!!

FEM for the Poisson problem in 2-D

The extension of the FEM for the Poisson equation in 2-D is analogous to the one-dimensional case. To this end, we need some definitions:

• Let $\Omega \subseteq \mathbb{R}^2$ (or \mathbb{R}^n , for any n), then we define:

$$L^2(\Omega) = \{ \text{functions } u \text{ such that } \int_{\Omega} u^2 dx \leq C < +\infty \}$$

$$\|u\|_{L^2(\Omega)} = \left(\int_{\Omega} u^2 dx \right)^{1/2}$$

furthermore: define:

$$(u, v)_{L^2} = \int_{\Omega} u(x)v(x) dx$$

The above (\cdot, \cdot) defines an inner product as:

$$(u, v) = (v, u)$$

$$(u, u) \geq 0$$

$$(u, u) = 0 \text{ , iff } u \equiv 0$$

$$H^1_0(\Omega) = \{ \text{functions } u \text{ such that: } \int_{\Omega} u^2 dx + \int_{\Omega} |\nabla u|^2 dx < C \}$$

$$\|u\|_{H^1_0(\Omega)} = \left(\int_{\Omega} u^2 dx + \int_{\Omega} |\nabla u|^2 dx \right)^{1/2}$$

Define:

$$(u, v)_{H^1_0(\Omega)} = \int_{\Omega} u(x)v(x) dx + \int_{\Omega} \langle \nabla u, \nabla v \rangle dx$$

$(\cdot, \cdot)_{H^1}$ also defines an inner product.

(57)

$H_0^1(\Omega) = \left\{ \text{functions } u \text{ such that } \int_{\Omega} |u_x|^2 dx \leq C \text{ and } u|_{\partial\Omega} = 0 \right\}$

$$\|u\|_{H_0^1} = \left(\int_{\Omega} |u_x|^2 dx \right)^{1/2}$$

$$(u, v)_{H_0^1} = \int_{\Omega} \langle \nabla u, \nabla v \rangle dx$$

$(\cdot, \cdot)_{H_0^1}$ defines an inner product also.

Remark: The function spaces $L^2(\Omega)$, $H^1(\Omega)$ and $H_0^1(\Omega)$ are examples of Hilbert spaces, i.e. complete inner product spaces.

2-D Poisson problem:

let $\Omega \subseteq \mathbb{R}^2$ be a domain, then the strong form of the Poisson problem with.

Poincaré inequality: $\forall u \in H_0^1(\Omega)$

$$\int_{\Omega} u^2 dx \leq C \int_{\Omega} |u_x|^2 dx \quad \text{--- (10)}$$

2-D Poisson problem:

(58)

For a domain $\Omega \subseteq \mathbb{R}^2$, the strong form of the Poisson equation is:

$$\begin{aligned} -\Delta u &= f \quad \text{on } \Omega \\ u|_{\partial\Omega} &= 0 \end{aligned} \quad \text{--- } \textcircled{11}$$

Variational formulation:

The weak form of $\textcircled{11}$ is: for $V = H_0^1(\Omega)$;

$\textcircled{12}$ Find a $v \in V = H_0^1(\Omega)$, ~~such that~~: find $u \in V$ such that

$$(u', v')_{H_0^1} = (f, v)_{L^2}$$

$$\text{or} \quad \int_{\Omega} \langle \nabla u, \nabla v \rangle dx = \int_{\Omega} f v dx.$$

~~we can check~~ A few remarks are in order.

1. if u is sufficient regular, then the strong form and the weak form are equivalent:

Proof: let u satisfy the strong form $\textcircled{11}$, for any $v \in H_0^1(\Omega)$, multiply both sides of $\textcircled{11}$ by v

$$-\Delta u v = f v.$$

Integrate both sides ~~by~~ over Ω

$$-\int_{\Omega} \Delta u v dx = \int_{\Omega} f v dx$$

Integrating by parts.

$$\int_{\Omega} \langle \nabla u, \nabla v \rangle dx = \int_{\partial \Omega} v \frac{\partial u}{\partial \nu} ds(x) = \int_{\Omega} f v dx$$

(59)

by the fact that $u \in H_0^1(\Omega) \Rightarrow u \equiv 0$ on $\partial \Omega$, we obtain:

$$\int_{\Omega} \langle \nabla u, \nabla v \rangle dx = \int_{\Omega} f v dx.$$

which is the ~~variational~~ variational formulation (12)

2. Define the energy functional:

$$J(\omega) := \frac{1}{2} \int_{\Omega} |\omega|^2 dx - \int_{\Omega} \omega f dx.$$

then $u \in H_0^1(\Omega)$ such that:

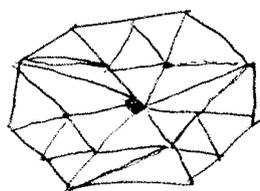
$$J(u) \leq J(\omega), \quad \forall \omega \in H_0^1$$

we can check that: u satisfies the variational formulation.

(12)

As in one space dimension, the FEM consists of solving the variational formulation on a finite dimensional subspace $V_h \subset V$.

Triangulations: In order to define this finite dimensional subspace, we consider a polygonal domain Ω (i.e. $\partial \Omega$ is a (possibly very large number of sides)), see figure 9:



+ Ω

figure 9

We consider a triangulation of Ω : T_h i.e.

(60)

$$\Omega = \bigcup_{k \in T_h} K = k_1 \cup k_2 \cup \dots \cup k_m$$

Here, k_i 's are non-overlapping triangles i.e. the no vertex of a triangle lies on the edge of another triangle - See figure 4.

Let:

$$h = \max_{k \in T_h} \text{diam}(k).$$

Now define the finite dimensional subspace V_h as:

$$V_h = \{v \text{ is continuous and } v|_K \text{ is linear with } v=0 \text{ on } \partial\Omega\}$$

i.e. continuous, piecewise linear functions that vanish on the boundary. It is easy to check that $V_h \in H_0^1(\Omega)$.

The resulting FEM is:

(14) Find $u_h \in V_h$ such that for all $v \in V_h$;

$$(u_h, v)_{H_0^1(\Omega)} = (f, v)_{L^2}.$$

$$\text{or } \int_{\Omega} \langle \nabla u_h, \nabla v \rangle dx = \int_{\Omega} f v dx.$$

Concrete realization.

As in the one-dimensional case, we obtain the following basis for V_h : Let $\{N_i\}$ denote the set of nodes of the triangles

in T_h , then define:

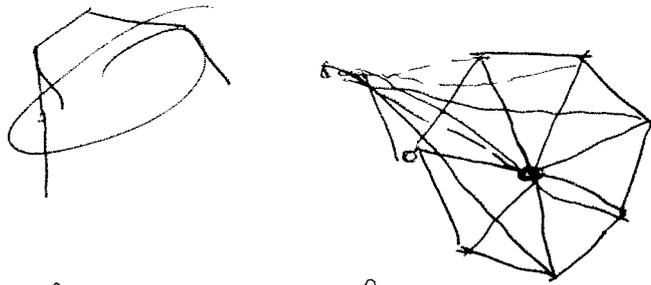
$$\Phi_j(x) \in V_h \text{ with}$$

$$\Phi_j(N_i) = 1 \quad \text{if } j=i$$

$$= 0, \text{ otherwise.}$$

(61)

These hat functions are shown in figure 5.



The support of each hat function Φ_j lies ~~in~~ on those triangles that have j as a vertex.

Each $v \in V_h$ can now be represented as:

$$v(x) = \sum_{j=1}^M n_j \Phi_j(x), \quad n_j = v(N_j), \quad \forall x \in \mathcal{R} \cup \mathcal{A}.$$

The FEM (14) can be formulated as (in the one-dimensional case)

$$\forall 1 \leq j \leq M \quad (m = \text{number of nodes})$$

$$(u_h, \Phi_j)_{H_0^1} = (f, \Phi_j)$$

$$\text{or} \quad \int \langle \sigma u_h, \nu \Phi_j \rangle dx = \int f \Phi_j dx \quad (15)$$

as u_h can also be represented as:

$$u_h = \sum_{i=1}^M u_i \Phi_i$$

\therefore from (15), we have: $1 \leq j \leq M$

$$\sum_{i=1}^M u_i \int \langle \sigma \Phi_i, \nu \Phi_j \rangle dx = \int f \Phi_j dx$$

The above can be reexpressed in the matrix form:

(6)

$$AU = F \quad \text{--- (16)}$$

with $U = \{u_i\}_{i=1}^M$

$$A = \{A_{ij}\}_{i,j=1}^M$$

$$A_{ij} = \int_{\Omega} \langle \nabla \phi_i, \nabla \phi_j \rangle dx$$

$$F = \{F_j\}_{j=1}^M \quad F_j = \int_{\Omega} f \phi_j dx.$$

Thus the FEM reduces to a matrix equation for the vector of unknowns U is obtained by inverting the stiffness matrix A and multiplying it with the load vector F .

As before, the matrix A is symmetric and positive definite. Hence the FEM (19) has a unique solution.

Rem: The matrix A is sparse as the hat functions are only supported on a few triangles.

Example: for simplicity, we let $\Omega = [0,1]^2$ (the unit square in \mathbb{R}^2) and consider the "uniform" triangulation as in figure 6.

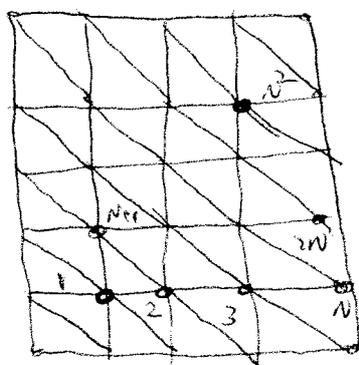


Figure 6.

Use the numbering as above.

