

FEM Error analysis for 2-D Poisson problem.

The exact solution of the 2-D Poisson equation satisfies:

$$(u, v)_{H_0^1} = (f, v)_{L^2}, \quad \forall v \in V = H_0^1$$

The ~~ft~~ clearly, $(u, v)_{H_0^1} = (f, v)_{L^2} \quad \forall v \in V_h \subset V$ — (1)

Here V_h is the finite dimensional FEM space.

The FEM solution satisfies,

$$(u_h, v)_{H_0^1} = (f, v)_{L^2} \quad \forall v \in V_h \quad \text{--- (2)}$$

Define the FEM error as: $e := u - u_h \in V$

Subtracting (2) from (1) and using the properties of the inner product, we obtain:

$$(e, v)_{H_0^1} \equiv 0, \quad \forall v \in V_h \quad \text{--- (3)}$$

The above property is Galerkin orthogonality of the error.

By definition:

$$\|e\|_{H_0^1}^2 = (e, e)_{H_0^1}$$

Now for any $v \in V_h$, define: $w = u_h - v \in V_h$:

by Galerkin orthogonality (3),

$$(e, w)_{H_0^1} = 0$$

$$\begin{aligned} \therefore \|e\|_{H_0^1}^2 &= (e, e)_{H_0^1} \\ &= (e, e)_{H_0^1} + (e, w)_{H_0^1} \\ &= (e, e+w)_{H_0^1} \\ &= (e, u-v)_{H_0^1} \leq \|e\|_{H_0^1} \|u-v\|_{H_0^1} \quad (\text{Cauchy-Schwarz}) \end{aligned}$$

Hence, $\|e\|_{H_0^1} \leq \|u-v\|_{H_0^1}, \quad \forall v \in V_h$

In other words,

(65)

$$\|e\|_{H_0^1} \leq \inf_{v \in V_h} \|u - v\|_{H_0^1} \quad \text{--- (9)}$$

As in the one-dimensional case, we can estimate the rhs of (9) by introducing the piecewise linear interpolant i.e.

Given $u \in H_0^1$, find a finite function $\tilde{u} \in V_h$ such that

$$u(N_i) = \tilde{u}(N_i), \quad \forall \text{ nodes of triangulation } T_h.$$

Clear and clearly (9) implies that:

$$\|e\|_{H_0^1} \leq \|u - \tilde{u}\|_{H_0^1}.$$

It turns out that if the triangles in T_h are not too irregular; then one can estimate that:

$$\|u - \tilde{u}\|_{H_0^1} \leq Ch \int_{\Omega} |\nabla u|^2 dx$$

$$\therefore \|e\|_{H_0^1} \leq Ch \int_{\Omega} |\nabla u|^2 dx$$

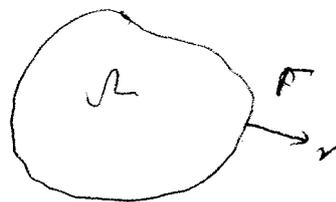
In particular, $\lim_{h \rightarrow 0} \|e\|_{H_0^1} = 0$ i.e., the method converges as the triangulation is refined.

A Poisson type Equation with different boundary conditions.

Let $\Omega \subseteq \mathbb{R}^2$ and $\Gamma = \partial\Omega$ be its boundary. We consider the following Poisson type equation:

$$-\Delta u + u = f \quad \text{in } \Omega$$

$$\textcircled{5} \quad \frac{\partial u}{\partial \nu} = g \quad \text{on } \Gamma$$



Here: $\frac{\partial \mathcal{U}}{\partial \nu}$ denotes the normal derivative:

$$\frac{\partial \mathcal{U}}{\partial \nu} = \nabla u \cdot \nu$$

with $\nu(x)$ being the unit outward normal at the point $x \in \Gamma$

The ~~pass~~ The boundary condition in the Poisson type problem is formed as the Neumann boundary condition and should be contrasted with the Dirichlet boundary condition that we have considered so far. It amounts to specifying the flux of the quantity u on the boundary Γ .

We are interested in ~~finding~~ designing a finite element method (FEM) to approximate the Neumann problem (5). To this end, we consider the following steps;

Step 1 \rightarrow Variational formulation.

It turns out that a suitable Dirichlet energy functional for (5) is:

$$J(u) := \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx + \frac{1}{2} \int_{\Omega} u^2 dx - \int_{\Omega} u f dx - \int_{\Gamma} \frac{\partial u}{\partial \nu} g(x) ds(x)$$

Note that $J(u)$ is well defined for

- $u \in H^1(\Omega)$
- $f \in L^2(\Omega)$
- $g \in L^2(\Gamma)$.

Here, we use the property that if $u \in H^1(\Omega)$, then $\frac{\partial u}{\partial \nu} \in L^2(\Gamma)$.

One can calculate using the principle of virtual work that for

u such that $J(u) \leq \inf_{w \in H^1(\Omega)} J(w)$, it satisfies

† $u \in H^1(\Omega)$, the following identity holds:

(57)

$$\int_{\Omega} \langle \sigma u, \sigma v \rangle dx + \int_{\Omega} u v dx = \int_{\Omega} f u dx + \int_{\Gamma} g v ds(x).$$

Denoting:

$$a(u, v) = (u, v)_{H^1(\Omega)} = \int_{\Omega} \langle \sigma u, \sigma v \rangle dx + \int_{\Omega} u v dx.$$

$$(f, u) = (f, u)_{L^2(\Omega)} = \int_{\Omega} f u dx$$

$$\langle g, v \rangle = (g, v)_{L^2(\Gamma)} = \int_{\Gamma} g v ds(x)$$

Then the suitable variational formulation for the Neumann problem (5) is: find $u \in H^1(\Omega)$

$$a(u, v) = (f, u) + \langle g, v \rangle, \quad \forall v \in H^1(\Omega). \quad (6)$$

Remarks

1. If u is sufficiently regular i.e. $D^2 u$ (the second derivative) of u exists and is in $L^2(\Omega)$, then by integration by parts:

$$\int_{\Omega} \langle \sigma u, \sigma v \rangle dx = - \int_{\Omega} v \Delta u dx + \int_{\Gamma} v \frac{\partial u}{\partial \nu} ds(x).$$

$$= \int_{\Omega} \dots$$

† Substituting the above in the variational formulation (6)

(68)

\therefore (6) \Rightarrow :

$$-\int_{\Omega} u \Delta u \, dx + \int_{\Gamma} u \frac{\partial u}{\partial \nu} \, ds(x) + \int_{\Omega} u v \, dx = \int_{\Omega} f v \, dx + \int_{\Gamma} g v \, ds(x)$$

$$\Rightarrow \int_{\Omega} (-\Delta u + u - f) v \, dx = \int_{\Gamma} \left(g - \frac{\partial u}{\partial \nu}\right) v \, ds(x), \quad \forall v \in H^1(\Omega) \quad \text{--- (7)}$$

Now, $H_0^1(\Omega) \subset H^1(\Omega)$

\therefore choosing $v \in H_0^1(\Omega)$, we know that $v \equiv 0$ on Γ

\therefore (7) reduces to:

$$\int_{\Omega} (-\Delta u + u - f) v \, dx = 0, \quad \forall v \in H^1(\Omega)$$

$$\Rightarrow -\Delta u + u = f \quad \text{in } \Omega:$$

Now substituting the above relation again in (7), for any $v \in H^1(\Omega)$,

we obtain:

$$\int_{\Gamma} \left(g - \frac{\partial u}{\partial \nu}\right) v \, ds(x) \equiv 0, \quad \forall v \in H^1(\Omega)$$

$$\Rightarrow \frac{\partial u}{\partial \nu} = g \quad \text{on } \Gamma$$

Thus if u is sufficiently regular, then solution of the variational formulation (6) is indeed a strong solution of the Neumann problem

(5).

2 • We ~~can~~ contrast the Dirichlet and Neumann boundary conditions. (69)

For the Dirichlet problem considered before, the boundary condition in the variational formulation was imposed by ~~app~~ restricting the variational formulation to functions in $H_0^1(\Omega)$ i.e. it was imposed on the function space. Such boundary conditions are termed as essential boundary conditions.

For the Neumann problem (5), the boundary condition was included in the variational formulation (6) itself. Such boundary conditions are termed as natural boundary conditions.

3. It is very difficult to impose a natural boundary condition for finite difference methods !!

Step 2 → Finite Element method:

For the FEM approximation of the Neumann problem (5), we restrict (6) to a finite dimensional subspace of ~~H^1~~ $U = H^1(\Omega)$. To this end.

Let $V_h \subset V$ be a finite dimensional subspace, then find $u_h \in V_h$ such that

$$(7) \quad a(u_h, v) = (f, v) + \langle g, v \rangle \quad \forall v \in V_h.$$

Choice of V_h : As before, let T_h be an admissible triangulation of a polygonal domain Ω .

$$T_h = \bigcup_j k_j, \quad k_j \text{'s are non-overlapping triangles.}$$

Then choose :

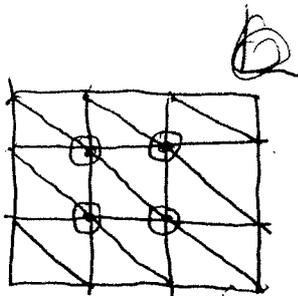
$$V_h = \{ v \text{ continuous and such that } v|_k \text{ is linear, for all triangles } k \in T_h \}$$

Note that we do not impose: $v \equiv 0$ on $\partial\Omega$ in the definition of the FEM space V_h . It is easy to check that $V_h \subset H^1(\Omega)$.

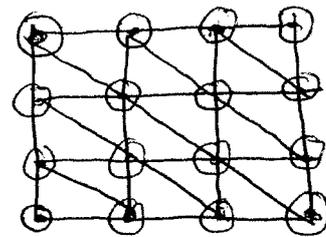
Furthermore, the hat functions ϕ_j considered before i.e.,

$$\begin{aligned} \phi_j(N_i) &= 1 \quad \text{if } i=j \\ &= 0, \text{ otherwise.} \end{aligned}$$

here $\{N_j\}$ also include boundary nodes, are a basis for V_h . (see figure 6).



Nodes for Dirichlet boundary conditions



Nodes for Neumann boundary conditions.

Figure 6

Step 3
Matrix formulation - Given that $\{\phi_j\}_j$ is a basis for V_h ,

we can reduce (6) to the following:

$$a(u, \phi_j) = (f, \phi_j) + \langle g, \phi_j \rangle \quad \forall j$$

Note that the second term in the rhs is only non-zero if $N_j \in \Gamma$. !!

as:
$$U = \sum_{i=1}^M u_i \phi_i$$

with M being the total number of nodes including boundary nodes.

by directness:

$$\sum_{i=1}^M u_i a(\phi_i, \phi_j) = (h, \phi_j) + (g, \phi_j), \quad \forall j$$

Hence (6) reduces to a matrix equation:

$$AU = F \quad (8)$$

with

$$U = \{u_i\}_{i=1}^M$$

$$F = \{f_i\}_{i=1}^M$$

$$f_j := (h, \phi_j) + (g, \phi_j)$$

$$A_{ij} = a(\phi_i, \phi_j) = \int_{\Omega} \underbrace{(\nabla \phi_i, \nabla \phi_j)}_{\underbrace{\quad}} dx + \int_{\Omega} \underbrace{\phi_i \phi_j}_{\underbrace{\quad}} dx$$

$\forall 1 \leq i, j \leq M$

Ex: Show that A is a symmetric, positive definite matrix and hence invertible.

Thus, FEM reduces to solving (8).

~~Error~~ Analy

Step 9: error analysis:

Exact soln: $a(u, v) = (f, v) + \langle g, v \rangle, \forall v \in V_n \subset V$

FEM: $a(u_h, v) = (f, v) + \langle g, v \rangle, \forall v \in V_n$

Definition of error: $e := u - u_h \in V$

Subtracting the above and using the definition of error;

Galerkin Orthogonality: $a(e, v) \equiv 0, \text{ for all } v \in V_n$

$w := u_h - v \in V_n, \text{ for } v \in V_n$

$$\begin{aligned}
\|e\|_{H^1(\Omega)}^2 &= (e, e)_{H^1} = a(e, e) \\
&= a(e, e) + a(e, w) \text{ (Galerkin orthogonality)} \\
&= a(e, e+w) \\
&= a(e, e+w) - a(e, u-v) \\
&= \int \langle \nabla e, \nabla(u-v) \rangle dx + \int e(u-v) dx \\
&\leq \cancel{\|e\|_{H^1}^2} \|e\|_{H^1} \|u-v\|_{H^1} \text{ (Cauchy-Schwarz)}
\end{aligned}$$

$\Rightarrow \|e\|_{H^1(\Omega)} \leq \|u-v\|_{H^1(\Omega)}, \forall v \in V_n \text{ --- } \textcircled{9}$

As before, we can define: $\tilde{u} \in V_n$ as the piecewise linear interpolant of u , i.e.

$\tilde{u}(v_i) = u(v_i), \forall v_i, 1 \leq i \leq m$

Then for regular triangulations, one can show that

$$\|u - \tilde{u}\|_{H^1} \leq h \|D^2 u\|_2$$

(73)

\therefore the final error estimate from (9) is

$$\|e\|_{H^1} \leq h \|D^2 u\|_2 \rightarrow 0 \text{ as } h \rightarrow 0.$$
