

## Implementation of FEM

(74)

We describe a simple implementation of FEM for the Poisson type equation with Neumann boundary conditions (5) on a polygonal domain  $\Omega$  with boundary  $\Gamma$ .

In the FEM, we will work with variational formulation (7) i.e.

$$\forall v \in H_0^1(\Omega) = V \\ \forall v \in V_h, \text{ find } u_h \in V_h \text{ such that}$$

$$a(u_h, v) = (f, v) + \langle g, v \rangle \quad (7)$$

By choosing  $V_h$  to be span of hat functions, the discrete variational formulation (7) boils down to the matrix equation

(8) i.e.

$$AU = f$$

with

Unknowns (degrees of freedom):

$$U = \{u_i\}_{i=1}^M$$

Stiffness matrix:

$$A = \{A_{ij}\}_{i,j=1}^M \\ A_{ij} = a(\varphi_i, \varphi_j) = \int_{\Omega} \langle v \varphi_i, v \varphi_j \rangle dx + \int_{\Gamma} \varphi_i \varphi_j ds$$

Load vector:

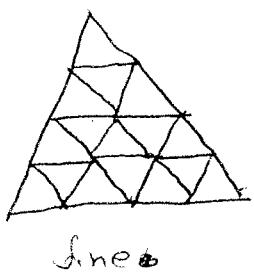
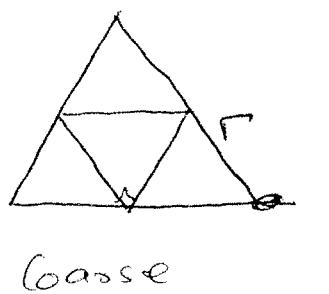
$$f = \{f_i\}_{i=1}^M \\ f_i = (f, \varphi_i) + \langle g, \varphi_i \rangle = \int_{\Omega} f \varphi_i dx + \int_{\Gamma} g \varphi_i ds(x).$$

Implementation of FEM has to undergo the following steps. (75)

### Step 1: Triangulation (Mesh Generation):

Given the polygonal domain  $\Omega$ , we need to divide into non-overlapping triangles. This is quite a sophisticated process and various efficient algorithms are available for this purpose.

A popular algorithm is to start with a coarse triangulation and then refine successively as shown in figure 10.



and so on .

figure 10 → Quasi-uniform mesh

Depending on this process results in the so-called quasi-uniform meshes i.e., triangles of "approximately" similar size.

Depending on the nature of the problem, we need to refine in some regions and remain coarse in others. Such meshes are typically adapted meshes, see figure 11

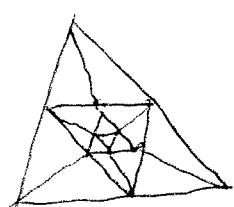
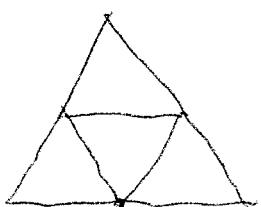


figure 11 → Adapted mesh

In any case, one relies on available free or commercial mesh generators. e.g. Distmesh (matlab), NETGEN, GLD, DELAUNOO etc. (76)

Assume that a triangulation of the domain  $\Omega$  is available, then the mesh information is usually stored in the following manner.

Let  $\{N_i\}_{i=1}^M$  be the number of nodes of the  $T_h$

$\{k_j\}_{j=1}^N$  be the number triangles in  $T_h$ ,

Then the mesh generator provides a numbering of to the nodes and the triangles, see figure 19.

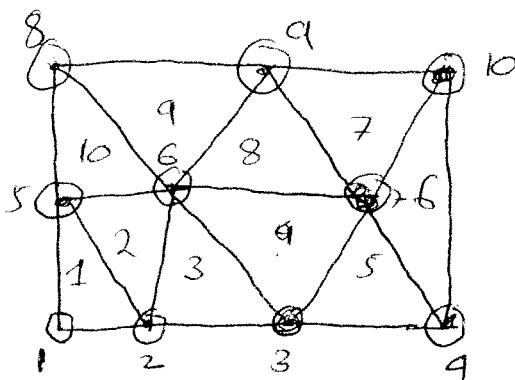


Figure 19

The following arrays are stored

1.  $Z(2, M)$  is the  $2 \times M$  (2 rows,  $m$  columns) Node array with

$Z(1, k)$ , referring to the Node  $N_k$  and

$Z(2, k)$ ,  $Z(2, k)$  representing the  $x$ - and  $y$ - coordinates

respectively of the node  $N_k$ .

2.  $T(3, N)$  is the  $3 \times N$  Triangle array with

$T(1, j)$  referring to the  $j$ th triangle  $k_j$

and

$T(i, j)$  ;  $i = 1, 2, 3$  are the numbers of the

three nodes  $\begin{matrix} \text{(vertices)} \\ \downarrow \end{matrix}$  corresponding to the triangle  $k_j$ .

(77)

As an example, the triangle array corresponding to figure 19 is:

$$\begin{bmatrix} 1 & 2 & 2 & 3 & 3 & 4 & 7 & 6 & 6 & 5 \\ 2 & 5 & 3 & 6 & 4 & 7 & 9 & 7 & 8 & 6 \\ 5 & 6 & 6 & 7 & 7 & 10 & 10 & 9 & 9 & 8 \end{bmatrix}$$

Note that once  $T(i, j)$  is called, the coordinates of the vertices is available by calling  $Z(., T(i, j))$ .

Most mesh generators provide  $Z$  and  $T$  and some additional information about the mesh. They try to efficiently number the nodes and triangles to minimize ~~bandwidth~~ band width of the resulting matrices.

Step 2. Building element stiffness matrices and element load vectors !!

From the structure of the global stiffness matrix,  $A$ , we

see that:

$$A_{ij} = \int_n \langle \partial \varphi_i, \partial \varphi_j \rangle dx + \int_n \varphi_i \varphi_j dx$$

$$= \sum_{k \in T_h} \int_{K_n} \langle \partial \varphi_i, \partial \varphi_j \rangle dx + \sum_{k \in T_h} \int_{K_n} \varphi_i \varphi_j dx$$

with  $K_n$  referring to the  $n^{th}$ -triangle in  $T_h$ . Thus,

$$A_{ij} = \sum_n A_{ij}^n$$

with

$$A_{ij}^n = \int_{k_n} \langle \varphi_i, \varphi_j \rangle dx + \int_{K_n} \varphi_i \varphi_j dx$$

Hence, the global

note that  $A_{ij}^n \neq 0$  if and only if

$N_i$  and  $N_j$  are both nodes of  $K_n$ .

Hence, the  $A_{ij}$  as  $k_n \in T_h$ , we know  $T(\alpha, n)$ ,  $\alpha=1,2,3$  are the numbers of the vertices of  $k_n$ .

The  $(x, y)$  coordinates of  $T(\alpha, n)$  are available from

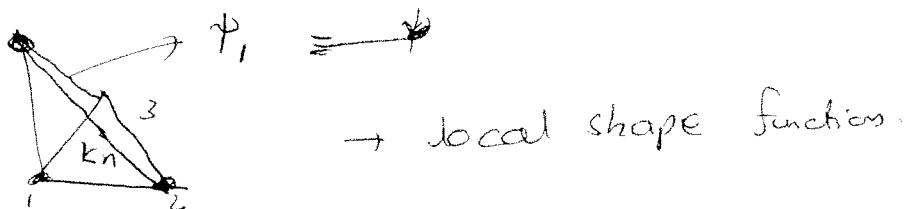
$$Z(i, T(\alpha, n)), i=1,2.$$

Once the co-ordinates are available, one can construct the so-called "local shape functions"  $\psi_\alpha$  ( $\alpha=1,2,3$ ) as

~~$\psi_\alpha$~~   $\psi_\alpha$  is linear function on  $k_n$  and

$$\begin{aligned} \psi_\alpha(N_{T(\beta, n)}) &= 1, \text{ if } \alpha = \beta \\ &= 0, \text{ otherwise, } \alpha, \beta = 1, 2, 3. \end{aligned}$$

See figure 18.



Then, we compute the element stiffness matrix as.

$$A_{\alpha\beta}^n = \int_{k_n} \left( (\nabla \psi_\alpha, \nabla \psi_\beta) + \psi_\alpha \psi_\beta \right) dx$$
$$\alpha, \beta = 1, 2, 3$$

Note that  $A_{\alpha\beta}^n$  is a  $3 \times 3$  Symmetric matrix.

The above integral is usually computed with a quadrature.

Similarly we can compute an element load vector.

$$f_\alpha^n = \int_{k_n} f \psi_\alpha dx + \int_{k_n \cap \Gamma} g \psi_\alpha ds(x).$$

Once we loop over each triangle  $n$  and calculate the element stiffness matrix  $A_{\alpha\beta}^n$  and element load vector  $f_\alpha^n$  ( $\alpha, \beta = 1, 2, 3$ ).

### Step 3. Assembly :-

This is the key step where the results of local computations (on each triangle) are assembled (combined) together to obtain the global stiffness matrix  $A$  and load vector  $f$ .

We use the following pseudo-code for assembly (MATLAB notation) for assembly.

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$$A = \text{zeros}(m, m)$$

$$f = \text{zeros}(m)$$

for  $n = 1, 2, \dots, N$  (do

for  $n = 1 : N$  (loop over all triangles)

fetch  $A^n = [A_{\alpha\beta}^n]$ ,  $f^n = (f_\alpha^n)$   $\alpha, \beta = 1, 2, 3$  and

$$A(T(\alpha, n), T(\beta, n)) = A(T(\alpha, n), T(\beta, n)) + A_{\alpha\beta}^n$$

$$F(T(\alpha, n)) = f(T(\alpha, n)) + f_\alpha^n$$

end.

Thus the 60 non-zero contributions of each triangle are assembled into the global stiffness matrix and global load vector.

#### Step 4. Solving the linear system.

Once  $A$  and  $f$  are available, the matrix equation

(8) can be solved either by a direct or an iterative method.

See numerical linear algebra textbooks.

Once the vector of unknowns  $U$  is determined, the FEM approximation  $u_h(x)$  is computed as:

$$u_h(x) = \sum_{i=1}^m u_i \varphi_i(x).$$

## FEM for abstract elliptic PDEs

(8)

In the previous chapter, we derived FEM for the Poisson problem, with both Dirichlet and Neumann BCs. To extend FEM to general elliptic PDEs, we use an abstract framework, that is built upon the concrete examples that we have seen.

As in FEM, the first step is a suitable variational formulation. To this end, we require a function space such that  $H^1$  or  $H_0^1$  to have the variational formulation in. We abstract this notion by introducing:

Let  $V$  be a Hilbert space i.e., vector space with an inner product  $(\cdot, \cdot)$  i.e.  $\forall v, w \in V$  then

$$\textcircled{1} \quad (v, w) = (w, v)$$

$$\textcircled{2} \quad (\alpha v + \beta w, co) = \alpha(v, co) + \beta(w, co)$$

$$\textcircled{3} \quad (v, v) = 0 \text{ iff } v=0 \text{ and } (v, v) \geq 0$$

In addition this inner product defines a norm by:

$$\|v\|_V^2 = (v, v)$$

If this inner product space is complete, then  $V$  is a Hilbert space.

Ex ①  $\mathbb{R}^n$  with standard scalar product

②  $\Omega \subseteq \mathbb{R}^n$ , then  $H_0^1(\Omega)$  is a Hilbert Space with  
 $(v, w) = \int_{\Omega} (v, w) dx$

③  $\Omega \subseteq \mathbb{R}^n$ , then  $L^2(\Omega)$  is a Hilbert space with

$$(v, w) = \int_{\Omega} vw dx$$

④  $\Omega \subseteq \mathbb{R}^n$ , then  $H^1(\Omega)$  is a Hilbert space with

$$(v, w) = \int_{\Omega} (v, w) + \nabla v \cdot \nabla w dx$$

• Bilinear form:

Let  $a: V \times V \rightarrow \mathbb{R}$  be such that:

$$a(\alpha v + \beta \bar{v}, \omega) = \alpha a(v, \omega) + \beta a(\bar{v}, \omega)$$

~~$$a(v, \alpha \omega + \beta \bar{\omega}) = \alpha a(v, \omega) + \beta a(v, \bar{\omega})$$~~

$\forall v, \bar{v}, \omega, \bar{\omega} \in V$

then  $a$  is defined as a bilinear form!

E.g.:  $(\cdot, \cdot)$  in a Hilbert space is a "symmetric" Bilinear form.

• Linear form

Let  $L: V \rightarrow \mathbb{R}$  be such that:

$$L(v) = L(\alpha v + \beta \bar{v}) = \alpha L(v) + \beta L(\bar{v}), \quad \forall v, \bar{v} \in V$$

then  $L$  is a linear form:

E.g. Let fix  $f \in L^2(\Omega)$ , then:

$$L_f(g) = \int_{\Omega} fg \, dx \text{ is a linear form.}$$

Now, we define the abstract variational problem:

(\*) Find  $u \in V$  such that:

$$a(u, v) = L(v), \quad \forall v \in V. \quad \text{--- (1)}$$

In general (1) may not be solvable, However under certain assumptions on  $a, L$ , we have the following result.

Theorem: let the bilinear form  $a$  have the following properties.

①  $a$  is symmetric

$$\text{i.e. } a(u, v) = a(v, u), \quad \forall u, v \in V$$

②  $a$  is continuous

$$\text{i.e. } |a(u, v)| \leq \gamma \|u\|_V \|v\|_V, \quad \text{for } \gamma > 0$$

③  $a$  is coercive i.e.  $\exists \alpha > 0$  such that

$$|a(v, v)| \geq \alpha \|v\|^2_V$$

let the linear form  $L$  satisfies.

④  $L$  is continuous i.e.

$$|L(v)| \leq \Lambda \|v\|_V, \quad \forall v \in V.$$

Then the variational problem ① has a unique solution  
 $\mathbf{u} \in V$ . Furthermore the solution satisfies the  
stability estimate:

$$\|u\|_V \leq \frac{\Lambda}{\alpha} \quad \text{--- (2)}$$

and ~~u also~~ and given the functional  $J$

$J: V \rightarrow \mathbb{R}$  with

$$J(v) = \frac{1}{2} a(v, v) - L(v)$$

then a solution ④ of ① satisfies:

$$J(u) \leq \inf_{v \in V} J(v).$$

Remarks

1. The existence of solution  $u$  of ① is a consequence of the Lax-Milgram Lemma.

2. Stability estimate:

as  $u \in V$

$$a(u, u) = L(u)$$

by coercivity;

$$\alpha \|u\|_V^2 \leq a(u, u) = L(u)$$

$$\leq \|L(u)\|$$

$$\leq \Delta \|u\|_V \quad (\text{continuity of } \ell)$$

$$\therefore \|u\|_V \leq \frac{\Delta}{\alpha} \quad . \text{ thus proving ②}$$

3. Uniqueness: Let  $u, \bar{u}$  be solutions of ①

$$\text{then: } a(u, v) = L(v)$$

$$a(\bar{u}, v) = L(v)$$

$$\therefore \text{Subtracting, } a(u, v) - a(\bar{u}, v) = \cancel{L(v)} 0$$

$$a(u - \bar{u}, v) = 0$$

$\therefore u - \bar{u}$  is a soln of ① with rhs 0, hence by

stability estimate ②

$$\|u - \bar{u}\|_V \leq 0 \Rightarrow u = \bar{u} \text{ and}$$

the solution is uniqueness

9. If  $u$  minimizes the "energy" functional  $J$  then,

$$\begin{aligned}
 0 = J'(u) &= \lim_{c \rightarrow 0} \frac{J(u+c\bar{v}) - J(u)}{c} \\
 &= \lim_{c \rightarrow 0} \frac{\frac{1}{2}a(u+c\bar{v}, u+c\bar{v}) - \frac{1}{2}a(u, u) + L(u+c\bar{v}) - L(u)}{c} \\
 &\stackrel{?}{=} \lim_{c \rightarrow 0} \frac{\frac{1}{2}c^2 a(\bar{v}, \bar{v}) + \frac{c}{2}a(u, \bar{v}) + \frac{c^2}{2}a(\bar{v}, \bar{v}) + cL(\bar{v})}{c} \\
 &= \lim_{c \rightarrow 0} a(u, \bar{v}) - L(\bar{v}) + \frac{c}{2}a(\bar{v}, \bar{v}) \quad (\text{by symmetry of } a) \\
 &= a(u, \bar{v}) - L(\bar{v}) \quad (\text{as } a(\bar{v}, \bar{v}) \leq \|A\bar{v}\|^2 < +\infty)
 \end{aligned}$$

$\therefore$  Hence,  $a(u, \bar{v}) = L(\bar{v})$ .

S. The proof of Lax-Milgram theorem works even if  $a$  is not-symmetric!!! Stability and Uniqueness do not require Symmetry. However, the energy principle relies on it

Abstract FEM: Given ①, let  $V_h \subset V$  be finite-dimensional subspace. Then FEM for ① amounts to:

Find  $u_h \in V_h$  such that  $\forall v \in V_h$

$$a(u_h, v) = L(v)$$

Let  $\{\varphi_j\}_{j=1}^m$  be a basis for  $V_h$ :

Then ③ reduces to:

Find  $u_h \in V_h$ , such that  $\forall \varphi_j \quad 1 \leq j \leq m$ ,

$$a(u_h, \varphi_j) = L(\varphi_j) \quad - ④$$

as  $u_h \in V_h$

$$u_h = \sum_{i=1}^M u_i \varphi_i$$

$\therefore$  (3) reduces to

$$\sum_{i=1}^M u_i a(\varphi_i, \varphi_j) = \text{④ } L(\varphi_j) \quad \text{if } 1 \leq j \leq M$$

Now define Stiffness matrix:

$$A = \{A_{ij}\}_{i,j=1}^M \quad A_{ij} = a(\varphi_i, \varphi_j)$$

load vector:

$$f = \{f_j\}_{j=1}^M \quad f_j = L(\varphi_j)$$

FEM reduces to the Matrix equation:

$$AU = F \quad \text{--- ⑤}$$

for the vector of unknowns:  $U = \{u_i\}_{i=1}^M$

Claim:  $A$  is symmetric and positive definite

Proof: for positive definiteness, let: vector  $v = u_j$  &  $w = \omega$

$$\begin{aligned} \therefore \langle \omega, Aw \rangle &= \sum_{i,j=1}^M \omega_j A_{ij} \omega_i \\ &= \sum_{i,j=1}^M \omega_j a(\varphi_i, \varphi_j) \omega_i \\ &= \sum_{i=1}^M a(\{\omega, \varphi_i\}, \{\omega, \varphi_i\}) \\ &= a(\bar{\omega}, \bar{\omega}) \geq \alpha \|\bar{\omega}\|^2 \end{aligned}$$

with  $\bar{\omega} = \sum_{i=1}^M \omega_i \varphi_i$

Thus,  $A$  is strictly positive definite, hence invertible!! (87)

### Concrete examples:

1. 2-D Poisson PDE with Dirichlet Boundary Conditions.

$$-\Delta u = f \text{ on } \Omega$$

$$u = 0 \text{ on } \Gamma$$

In this case,  $V = H_0^1(\Omega)$

$$a(v, w) = \int_{\Omega} \langle \nabla v, \nabla w \rangle dx$$

$$L(w) = \int_{\Omega} f w dx$$

### Exercises

(\*) Show that

- ①  $a$  is symmetric bilinear form
- ②  $a$  is continuous
- ③  $a$  is coercive.
- ④  $L$  is continuous

### Hints

(2) Cauchy-Schwarz and definition of  $\|u\|_{H_0^1}$

(3) Definition of  $\|u\|_{H_0^1}$

(4) Cauchy-Schwarz and Poincaré inequality.

2. 2-D Poisson PDE with Neumann boundary conditions.

$$-\Delta u + u = f \text{ in } \Omega$$

$$\frac{\partial u}{\partial \nu} = g \text{ in } \Gamma$$

In this case,  $\bullet V = H^1(\Omega)$

$$a(v, w) = \int_{\Omega} \langle \nabla v, \nabla w \rangle dx + \int_{\Omega} vw dx$$

$$L(v) = \int_{\Omega} fv dx + \int_{\Gamma} g v ds(x)$$

Ex: Show that the above formulation satisfies the hypothesis of Theorem 1.

3. A one-dimensional bi-harmonic equation:

$$\bullet u'''' = f \text{ in } (0, 1)$$

$$u(0) = u(1) = u'(0) = u'(1) = 0$$

Here  $u'''' = \frac{d^4 u}{dx^4}$ ,

~~In this~~ Define  ~~$H^2(\Omega)$~~  =  $\int$

Define  $\underline{H^2(\Omega)} = H^2(0, 1) = \left\{ v : \begin{array}{l} \int v^2 dx < +\infty \\ \int |v'|^2 dx < +\infty \\ \int |v''|^2 dx < +\infty \end{array} \right\}$

Ex: Check that  $H^2(0,1)$  is a Hilbert space with inner product:

$$(u, \omega) = \int_0^1 u\omega \, dx + \int_0^1 u' \omega' \, dx + \int_0^1 u'' \omega'' \, dx.$$

and norm  $\|u\|_{H^2}^2 = \int_0^1 u^2 \, dx + \int_0^1 |u'|^2 \, dx + \int_0^1 |u''|^2 \, dx$

Now define a subspace of  $H^2(0,1)$  as

$$H_0^2(0,1) = \{u \in H^2(0,1) \text{ with } u(0) = u'(0) = u(1) = u''(1) = 0\}$$

Check that  $H_0^2$  is a Hilbert space with some norm!

Then define,  $a(u, v) = \int_0^1 u'' v'' \, dx$

$$L(v) = \int_0^1 fv \, dx$$

clearly,  $a$  is symmetric, check that  $a$  is continuous.

Furthermore, as  $u(0) = u(1) = 0$ , we have the Poincaré inequality;

$$\int_0^1 u^2(x) \, dx \leq \int_0^1 |u'|^2 \, dx \quad \text{--- (6)}$$

Let  $\omega \in H_0^2(0,1)$ , define  $\omega = u'$ .

clearly  $\omega(0) = \omega(1) = 0$ , by applying the

Poincaré inequality to  $\omega$ , we obtain,

$$\int_0^1 |\omega|^2 \, dx \leq \int_0^1 |u'|^2 \, dx$$

$\therefore$  for  $u \in H_0^2(0,1)$

$$\int_0^1 u^2 dx \leq \int_0^1 |u'|^2 dx \leq \int_0^1 |u''|^2 dx$$

(90)

Hence:

$$a(v, v) = \int_0^1 |u''|^2 dx$$

but:  $\|v\|_{H^2}^2 = \int_0^1 v^2 + \int_0^1 |v'|^2 + \int_0^1 |v''|^2$

$$\leq 3 \int_0^1 |u''|^2 dx \leq 3 a(v, v)$$

$$\therefore \text{Hence } a(v, v) \geq \frac{1}{3} \|v\|_{H^2}^2$$

so,  $a$  is coercive with  $\alpha = \frac{1}{3}$ .

Check that  $L$  is continuous.

Therefore, the conditions of Theorem ① apply and we can build the following FEM for the 1-D "biharmonic" equation, find let  $V_h$  be a finite dimensional subspace of  $H_0^2(0,1)$

find  $c_h \in V_h$  such that  $\forall v_h \in V_h$ :

$$\int_0^1 d_h^{(1)} v_h'' dx = \int_0^1 f v_h dx$$

(91)

Example 4: A 2-D convection-Diffusion problem.

Let  $\Omega \subseteq \mathbb{R}^2$ , then consider define the convection-diffusion equation:

$$-\mu \Delta u + (\beta \cdot \nabla) u = f \text{ in } \Omega$$

$$u = 0 \text{ on } \Gamma$$

In other words,  $u = u(x_1, x_2)$   $\beta = (\beta_1, \beta_2)$ , then

$$-\mu(u_{xx} + u_{yy}) + \beta_1 u_x + \beta_2 u_y = f \text{ in } \Omega$$

$$u = 0 \text{ on } \Gamma \quad (6)$$

$\mu$  scales the diffusion and  $\beta$  is the direction of convection.

Let  $V = H_0^1(\Omega)$ :

$$a(v, w) = \int_{\Omega} \langle \nabla v, \nabla w \rangle dx + \int_{\Omega} \beta_1 v_x w dx + \int_{\Omega} \beta_2 v_y w dx$$

Note that

(1)  $a: V \times V \mapsto \mathbb{R}$  is well defined.

(2)  $a$  is bilinear and  $a$  is NOT symmetric.

(3)  $a$  is continuous as:

$$\begin{aligned} |a(v, w)| &\leq \left| \int_{\Omega} \langle \nabla v, \nabla w \rangle dx \right| + \beta_1 \int_{\Omega} |v_x w| dx + \beta_2 \int_{\Omega} |v_y w| dx \\ &\leq \left( \int_{\Omega} |v_x|^2 dx \right)^{1/2} \left( \int_{\Omega} |w|^2 dx \right)^{1/2} + |\beta_1| \left( \int_{\Omega} v_x^2 dx \right)^{1/2} \left( \int_{\Omega} w^2 dx \right)^{1/2} \\ &\quad + |\beta_2| \left( \int_{\Omega} v_y^2 dx \right)^{1/2} \left( \int_{\Omega} w^2 dx \right)^{1/2} \\ &\leq \|v\|_{H_0^1} \|w\|_{H_0^1} \end{aligned}$$

Note we use the Poincaré inequality and

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$$C = \max(|\beta_1|, |\beta_2|)$$

④ a is coercive as:

$$\begin{aligned} \text{After } a(u, v) &= \int_{\Omega} |uv|^2 dx \\ &\quad + \int_{\Omega} \beta_1 u v_{x_1} dx + \int_{\Omega} \beta_2 v v_{x_2} dx \\ &\geq \int_{\Omega} |uv|^2 dx + \beta_1 \int_{\Omega} \left(\frac{v^2}{2}\right)_{x_1} dx + \beta_2 \int_{\Omega} \left(\frac{v^2}{2}\right)_{x_2} dx \\ &= \int_{\Omega} |uv|^2 dx \quad (\text{as } v \equiv 0 \text{ on } \Gamma) \end{aligned}$$

Define the linear form:

$$L(v) = \int_{\Omega} fv dx$$

L is clearly continuous.

\* ∵ the conditions of the non-symmetric version of the Lax-Milgram theorem apply and we can build the following FEM for the convection-diffusion equation,

Let  $V_h$  be a finite dimensional subspace of  $H_0^1$

Find  $u_h \in V_h$  such that

$$\int_{\Omega} \langle \alpha u_h, v \rangle dx + \int_{\Omega} \beta_1 u_h v_{x_1} dx + \int_{\Omega} \beta_2 u_h v_{x_2} dx$$

$$= \int_{\Omega} fu \, dx \quad \forall u \in V_h$$

(93)

However, the resulting matrix will not be symmetric.  
though still invertible !!!

Abstract FEM error estimate:

let  $\mathbb{B}$   $V$  be a Hilbert space and  $a$  being a continuous, coercive bilinear form and  $L$  a continuous linear form, then  $\exists$   $u$  solution

$$\forall v \in V, \quad a(u, v) = L(v) \quad \text{--- (1)}$$

The FEM for (1) is: let  $V_h \subset V$  be a finite dimensional subspace, then find  $u_h \in V_h$  such that

$$\forall v \in V_h, \quad a(u_h, v) = L(v) \quad \text{--- (2)}$$

Let the error be defined as;

$$e := u - u_h \in V$$

$$\text{clearly: } a(u, v) = L(v), \quad \forall v \in V_h \subset V \quad \text{--- (3)}$$

Subtracting (2) from (3) and using the linearity of  $a$ ,

$$(4) \quad a(e, v) = 0, \quad \forall v \in V_h$$

The above property is Galerkin - Orthogonality.

& for any  $v \in V_h$  define

$$\omega = u_h - v \in V_h$$

$$\therefore \text{from (4), } \cancel{a(e, v)} \quad a(e, \omega) = 0$$

We have:

(99)

$$\begin{aligned}\|e\|_V^2 &\leq \frac{1}{\alpha} a(e, e) \quad (\text{Coercivity}) \\ &\leq \frac{1}{\alpha} (a(e, e) + a(e, \omega)) \quad (\text{Galerkin orthogonality}) \\ &\leq \frac{1}{\alpha} (a(e, e+\omega)) \quad (\text{Linearity}) \\ &\leq \frac{1}{\alpha} (a(e, u-v)) \quad (\text{Definition}) \\ &\leq \frac{1}{\alpha} |a(e, u-v)| \\ &\leq \frac{\gamma}{\alpha} \|e\|_V \|u-v\|_V\end{aligned}$$

$\therefore \forall v \in V_h$ ,

$$\|e\|_V \leq \frac{\gamma}{\alpha} \|u-v\|_V \quad -\textcircled{7}$$

One can choose:  $\tilde{u}$  to be piecewise linear interpolant of  $u$ , then if the triangles are shape-regular, we have:

$$\text{If } v = H^1(\Omega), \text{ then } \|u-v\|_{H^1(\Omega)} \leq Ch \|u\|_{H^2(\Omega)}$$

In these cases, the abstract error estimate is

$$\begin{aligned}\|e\|_H &\leq \frac{C\gamma}{\alpha} h \|u\|_{H^2(\Omega)} \\ &\rightarrow 0 \text{ as } h \rightarrow 0.\end{aligned}$$

In fact, if one uses a piecewise quadratic interpolant  $\hat{u}$ , then

$$\|u-\hat{u}\|_{H^1(\Omega)} \leq Ch^2 \|u\|_{H^3(\Omega)}$$

Thus, if the finite dimensional subspace  $V_h$  consists of continuous, piecewise quadratic functions, then. (95)

$$\begin{aligned}\|e\|_{H^1} &\leq \frac{\gamma}{\lambda} \|u - \hat{u}\|_{H^1} \\ &\leq \frac{\gamma h^2}{\lambda} \|u\|_{H^3} \quad - (8)\end{aligned}$$

Thus if the exact solution  $u$  is sufficiently regular (which is the case for the Poisson problem with  $f \in H^1(\Omega)$ ), then (8) implies a higher rate of convergence !!!

### Piecewise Quadratic Finite Elements :

let  $T_h$  be a triangulation of  $\Omega$ . Then defines:

$$P_2(k) = \left\{ \text{All functions } v \text{ such } v \text{ is quadratic on } k \right\}$$

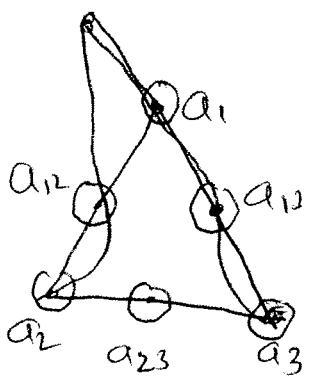
$\forall k \in T_h$ :

Define the finite dimensional - Subspace:

$$V_h = \left\{ v \in P_2(k), \forall k \in T_h \text{ and } v \in C^0(\bar{\Omega}) \right\}$$

Then one can check that  $V_h \subset H^1(\Omega)$

Furthermore, a basis can be obtained by considering the following local shape functions. Let  $k$  - let the triangle  $k$  be such that: its vertices are  $a_1, a_2, a_3$ ; let  $a_{12}, a_{13}, a_{23}$  be the mid-points of edges spanning  $a_1$  and  $a_2$ ,  $a_2$  and  $a_3$ ,  $a_1$  and  $a_3$  respectively. See figure 16.

Figure 16

Consider functions:  $\psi$  such that:

$\psi$  is quadratic on  $\mathcal{E}$  and

$$\psi(a_1) = 1$$

$$\psi(a_{12}), \psi(a_{13}), \psi(\underline{a_2}) \sim \psi(a_{23}), \psi(a_2), \psi(a_3) = 0$$

Similarly, one can define shape-functions defined on each node and edge midpoint (see Figure 16)

The above  $\psi$ 's form a global basis when they are extended.