

Parabolic Equations:

The Heat (diffusion) Equation is a prototype for parabolic equations. In several space dimensions, the heat equation is:

$$U_t - \Delta U = 0 \quad \text{in } \Omega \times [0, T)$$

$$U(x, 0) = U_0(x) \quad \text{--- (1)}$$

The above equation has to be supplemented by suitable boundary conditions.

~~for~~ We start with the one-dimensional case and consider the heat equation in one space dimension;

$$U_t - U_{xx} = 0 \quad \text{in } (0, 1) \times (0, T)$$

$$U(x, 0) = U_0(x) \quad \text{in } (0, 1) \quad \text{--- (2)}$$

$$U(0, t) = U(1, t) = 0 \quad \text{on } \overline{(0, 1) \times (0, T)}.$$

⊗ We consider Dirichlet boundary conditions here.

Exact solution formulas.

To find an exact solution formula, we have the following ansatz (separation of variables):

$$U(x, t) = X(x) T(t)$$

$$\text{Then } U_t = T'(t) X(x)$$

$$U_{xx} = T(t) X''(x)$$

∴, the PDE in (2) reduces to:

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$$\textcircled{1} \quad T'(t) X(x) = T(t) X''(x)$$

$$\Rightarrow \frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)} \quad \text{--- (3)}$$

Note that the lhs of above expression is a function of time t . Similarly, the rhs is a function of the space variable x .

The only (3) can be true for $\forall x \in (0,1)$ and $\forall t \in (0,\infty)$ if

$$\frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)} = -\lambda_k \text{ (Some constant)} \quad \text{--- (4)}$$

From (4) $X''(x) + \lambda_k X(x) = 0$ in $(0,1)$

$$X(0) = X(1) = 0 \quad \text{--- (5)}$$

Note that (5) has a general solution (it can be easily verified)

$$X_k(x) = \sin(k\pi x), \quad \lambda_k = (k\pi)^2 \quad \text{--- (6)}$$

$\forall k \in \mathbb{Z}$ (Integers).

Hence from (5) (4), we obtain:

$$T'(t) = -(k\pi)^2 T(t)$$
$$\Rightarrow T(t) = e^{-(k\pi)^2 t} \quad \text{--- (7)}$$

Plugging (6) and (7) into the variable separable ansatz, we

obtain that:

$$u(x,t) = X(x)T(t) = e^{-(k\pi)^2 t} \sin(k\pi x)$$

is a general solution of:

$$u_t - u_{xx} = 0 \quad \text{in } (0,1) \times (0,\tau)$$

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$$u(x,0) \quad u(0,t) = u(1,t) = 0$$

In order to obtain a unique particular solution, we observe that

$$\textcircled{1} \quad \int_0^1 \sin(k\pi x) \sin(m\pi x) dx = \begin{cases} 0 & \text{if } k \neq m \\ \frac{1}{2} & \text{if } k = m. \end{cases}$$

Hence: $\{\sin(k\pi x)\}_{k \in \mathbb{Z}}$ is a basis in $L^2(0,1)$ (Fourier-Sine Series):

\therefore for any $f \in L^2(0,1)$, we have the Fourier expansion,

$$f = \sum_{k=-\infty}^{+\infty} f_k \sin(k\pi x)$$

$$\text{with: } f_k = 2 \int_0^1 f(x) \sin(k\pi x) dx$$

Hence for any initial data u_0 in $\textcircled{2}$ such that:

$$u_0 \in L^2(0,1),$$

$$\text{By Fourier expansion; } u_0(x) = \sum_{k=-\infty}^{+\infty} u_k^0 \sin(k\pi x).$$

$$\text{with } u_k^0 = 2 \int_0^1 u_0(x) \sin(k\pi x) dx.$$

The convergence of the above infinite summation is a consequence of Fourier series.

Check that;

$$u(x,t) = \sum_{k=-\infty}^{+\infty} u_k^0 e^{-(k\pi)^2 t} \sin(k\pi x) \quad \text{--- (8)}$$

is a solution of the Heat-equation (2).

Proof: $u_t = \sum_{k=-\infty}^{+\infty} -(k\pi)^2 u_k^0 e^{-(k\pi)^2 t} \sin(k\pi x)$

$$u_{xx} = \sum_{k=-\infty}^{+\infty} (k\pi)^2 u_k^0 e^{-(k\pi)^2 t} \sin(k\pi x)$$

$$\therefore u_t = u_{xx}$$

~~Use~~ $u(0,t) = u(l,t) = 0$.

$$u(x,0) = \sum_{k=-\infty}^{+\infty} u_k^0 \sin(k\pi x) = u_0(x)$$

In fact one can show that (8) is the unique solution of the Heat equation (2).

Evaluation of the exact solution formula (8)

we use the following algorithm:

Given $u_0(x)$

step 1: Expand ~~(8)~~ u_0 in the truncated fourier series;

$$u_0^N(x) = \sum_{k=-N}^N u_k^0 \sin(k\pi x)$$

The error $|u_0 - u_0^N|$ is small for large N if u_0 is a smooth function.

to calculate u_0^t , we use a quadrature to approximate:

$$u_0^t = 2 \int_0^1 u_0(x) \sin(k\pi x) dx$$

Step 2: The approximate solution (8) for the formula (8) is:

$$u_N = \sum_{k=-N}^N u_0^k e^{-(k\pi)^2 t} \sin(k\pi x).$$

Note that we have 2 sources of errors:

(1) Error due to finite-truncation of Fourier

Sine series

(2) Error due to quadrature.

Given this, the exact solution formula can as well be replaced by a numerical method.

In order to design suitable numerical methods, we will need some qualitative properties of a solution of the Heat equation (2).

2. Energy method let u be a solution of (2)

Define the Energy:

$$E(t) := \frac{1}{2} \int_0^1 u^2(x,t) dx$$

Differentiate E w.r.t time.

$$\begin{aligned} E'(t) \frac{dE}{dt} &= \frac{1}{2} \int_0^1 (u^2)_t dx \\ &= \int_0^1 u u_t dx \end{aligned}$$

$$\begin{aligned}
 &= \int_0^1 u u_{xx} dx \quad (\text{from } \textcircled{2}) \\
 &= - \int_0^1 u_x^2 dx + u u_x \Big|_0^1 \quad (\text{Integration by parts}) \\
 &= - \int_0^1 u_x^2 dx \quad (\text{from Boundary conditions}).
 \end{aligned}$$

$$\therefore \frac{dE}{dt} = - \int_0^1 u_x^2 dx \leq 0$$

$$\Rightarrow E(t) \leq E(0), \quad \forall t \in (0, T] \quad \textcircled{a}$$

In other words, the energy of the solution decreases in time:

Consequence of energy method

Uniqueness: Let u, \bar{u} be 2 solutions of $\textcircled{2}$

$$\text{let } w := u - \bar{u}$$

clearly w satisfies the following heat equation;

$$w_t = w_{xx} \quad \text{in } (0,1) \times (0,T)$$

$$w(0,t) = w(1,t) = 0$$

$$w_0 = w(0,x) \equiv 0$$

Hence, define:
$$\bar{E}(t) = \frac{1}{2} \int_0^1 w^2(x,t) dx.$$

by the energy method, (9),

$$\bar{E}(t) \leq \bar{E}(0)$$

$$\Rightarrow \int_0^1 \omega^2(x,t) dx \leq \int_0^1 \omega^2(x,0) dx, \quad \forall t$$

$$\Rightarrow \int_0^1 \omega^2(x,t) dx \leq 0$$

$$\Rightarrow \omega(x,t) \equiv 0.$$

$$\Rightarrow u(x,t) = \bar{u}(x,t), \quad \forall t$$

and the solution of (2) is unique.

3. Maximum principles

Observe that ~~the~~ a bound on the energy E implies a bound on the L^2 -norm of the solution at any given time t as:

$$\sup_{0 \leq t \leq T} \|u(\cdot, t)\|_{L^2} \leq \sup \|u_0\|_{L^2}$$

Maximum principles provide bounds on the L^∞ (maximum) norm. We have the following statement:

Lemma (maximum principle), let u be a solution of the heat equation (2), then:

$$\min(0, u_0(x)) \leq u(x,t) \leq \max(0, u_0(x)) \quad (10)$$
$$\forall x \in [0,1], \quad \forall t \in [0,T].$$

Pf: First, (10) implies that the maximum (minimum)

of the solution of heat equation is attained on the parabolic boundary i.e. initial line or two side boundaries. See figure 1.

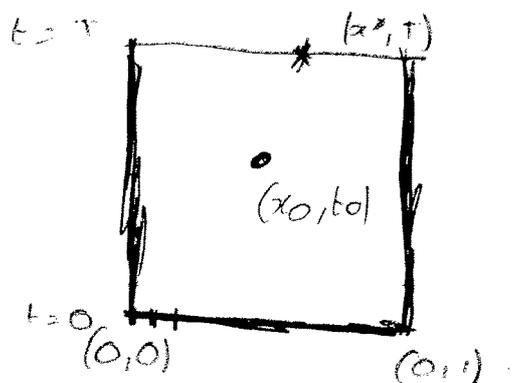


Figure 1.

Domain for the heat equation
Parabolic boundary is shaded.

We will only focus on the maximum principle. The minimum principle can be proved analogously.

We proceed by contradiction.

First, assume that (x_0, t_0) is a strict maximal point of u with $0 < x_0 < 1$, $0 < t_0 < T$ (Figure 1)

i.e. $u(x_0, t_0) > u(x, t), \quad \forall x \in (0, 1), t \in (0, T)$

Clearly: $u_t(x_0, t_0) \equiv 0$
 $u_{xx}(x_0, t_0) < 0$ (maximal point).

Hence, $u_t(x_0, t_0) - u_{xx}(x_0, t_0) > 0$

This is a contradiction as u solves the heat equation (2),
So maximum cannot be attained at in the interior.

Next, let maximum be attained at (x^*, T) (see figure 1) lying at the final time.

clearly $u_{xx}(x^*, T) > 0$

also, $u_t(x^*, T) = \lim_{h \rightarrow 0} \frac{u(x^*, T) - u(x^*, T-h)}{h} > 0$

Hence: $u_t(x^*, T) - u_{xx}(x^*, T) > 0$

which contradicts (2).

Thus, a strict maximum cannot be attained at the final time. Hence, it can only be attained on the parabolic boundary.

Now consider ~~$u^\epsilon(x, t) = u(x, t) + \epsilon t$ (for some $\epsilon > 0$)~~

~~clearly: $u_t^\epsilon = u_t + \epsilon$~~

~~and $u_{xx}^\epsilon = u_{xx}$~~

~~Let (x_0, t_0) be a maximum (not a strict maximum) of u .~~

~~$\therefore u_t(x_0, t_0) \leq 0 \Rightarrow u_t^\epsilon(x_0, t_0) = \epsilon$~~

~~$u_{xx}(x_0, t_0) < 0$~~

~~Hence, $u^\epsilon(x_0, t_0)$~~

Now consider, $u^\epsilon(x, t) = u(x, t) - \epsilon t$, for $\epsilon > 0$

Assume that for a fixed ϵ , u^ϵ attains a maximum at (x_0, t_0)

$\therefore 0 = u_t^\epsilon(x_0, t_0) = u_t - \epsilon$

$\Rightarrow u_t(x_0, t_0) = \epsilon > 0$

Similarly: $u_t(x_0, t_0) \leq 0$

as: $u_{xx}^\epsilon(x_0, t_0) = u_{xx}(x_0, t_0)$

we have $u_{xx}(x_0, t_0) \leq 0$

hence, $u_t(x_0, t_0) - u_{xx}(x_0, t_0) \geq \epsilon > 0$

This contradicts (2) as u is a solution of the.

Heat equation.

A similar argument can be carried out for the point (x^*, t)

hence: $\max_{\substack{0 \leq x \leq 1 \\ 0 \leq t \leq T}} u^\epsilon(x, t) \leq \max(0, u_0^\epsilon(x))$

or $\max_{\substack{0 \leq x \leq 1 \\ 0 \leq t \leq T}} u(x, t) \leq \max(0, u_0(x))$

as the rhs is independent of ϵ , we let $\epsilon \rightarrow 0$ in the lhs and obtain

$$\max_{\substack{0 \leq x \leq 1 \\ 0 \leq t \leq T}} u(x, t) \leq \max(0, u_0(x))$$

Thus proving the maximum principle.

• Finite difference schemes for the Heat Equation.

Consider the heat equation:

$$\begin{aligned}
 & u_t = u_{xx} \text{ on } (0,1) \times (0,T] \\
 \textcircled{1} \quad & u(0,t) = u(1,t) = 0, \quad \forall t \in (0,T] \\
 & u(x,0) = u_0(x), \quad \forall x \in (0,1)
 \end{aligned}$$

In order to approximate the solutions of the heat equation, we derive a finite difference scheme in the following steps:

Step 1: Discretizing the domain:

We divide $(0,1)$ into N equally spaced intervals of grid size Δx by defining:

$$\begin{aligned}
 x_0 &= 0 \\
 x_j &= j \Delta x \quad \text{with} \quad \Delta x = \frac{1}{N+1} \\
 x_{N+1} &= 1
 \end{aligned}$$

Similarly we divide $(0,T)$ into M equally spaced intervals of time step $\Delta t > 0$ by:

$$\begin{aligned}
 t_0 &= 0 \\
 t^n &= n \Delta t \quad \text{with} \quad \Delta t = \frac{T}{M+1} \\
 t^{M+1} &= T
 \end{aligned}$$

The resulting grid is shown in figure 1.

Step 2: Discretizing the solution u .

The point-values of the solution u on the grid are approximated as:

$$\textcircled{2} \quad v_j^n \approx u(x_j, t^n).$$

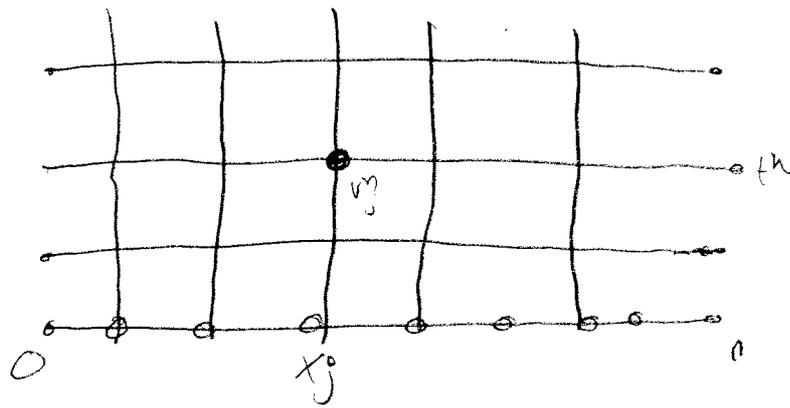


Figure 1

Step 3: Discretizing the derivatives:

The derivatives in (1) need to be replaced by finite differences.

(a) Spatial derivative: As for the Laplace equation, the spatial derivative is approximated with a central difference: i.e.,

$$u_{xx}(x_j, t_n) \approx \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2}, \quad \forall (n, j).$$

(b) Time derivative: Here we have some choices, the simplest choice to approximate the time derivative is to use a forward difference i.e.;

$$u_t(x_j, t_n) \approx \frac{v_j^{n+1} - v_j^n}{\Delta t}$$

Step 4: The finite difference scheme:

The approximation of the heat equation is;

$$\frac{V_j^{n+1} - V_j^n}{\Delta t} + \frac{V_{j+1}^n - 2V_j^n + V_{j-1}^n}{\Delta x^2} = 0 \quad \text{--- (2)}$$

$\forall 1 \leq j \leq N; 0 \leq n \leq m$

or:

$$V_j^{n+1} = (1 - 2\lambda) V_j^n + \lambda V_{j+1}^n + \lambda V_{j-1}^n$$

with $\lambda = \frac{\Delta t}{\Delta x^2}$

with: $V_j^0 = u(x_j, 0) = u_0(x_j), \quad \forall 1 \leq j \leq N$ (Initial conditions)

and $V_0^n \equiv 0, \quad \forall 1 \leq n \leq m$

$V_{N+1}^n \equiv 0, \quad \forall$

We introduce the following notations:

forward difference in space: $D_x^+ w_j^n = \frac{w_{j+1}^n - w_j^n}{\Delta x}$

backward difference: $D_x^- w_j^n = \frac{w_j^n - w_{j-1}^n}{\Delta x}$

forward difference in time: $D_t^+ w_j^n = \frac{w_j^{n+1} - w_j^n}{\Delta t}$

backward difference: $D_t^- w_j^n = \frac{w_j^n - w_j^{n-1}}{\Delta t}$

Hence (2) can be recast as:

~~$$D_t^+ V_j^n - D_x^- D_x^+ V_j^n = 0 \quad \text{--- (3)}$$~~

$\forall 1 \leq j \leq N, 0 \leq n \leq m$

The implementation of the finite difference scheme (2) is straightforward

Given values $\{V_j^n\}_{1 \leq j \leq N}$, compute $\{V_j^{n+1}\}_{1 \leq j \leq N}$ from the update formula (3) and the boundary conditions.

Name

Terminology: The scheme (2) or (3) is termed as Explicit finite difference scheme as the time stepping is explicit @ (forward) Euler.

Numerical results. See slides.

The numerical results strongly depend on the choice of time step or choice of the parameters \rightarrow :

$$\tau := \frac{\Delta t}{\Delta x^2}$$

In order to understand these results, we need to perform stability analysis.

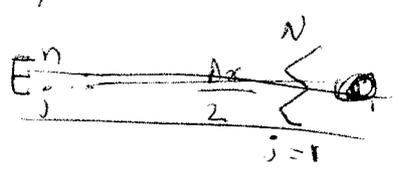
Energy stability: @ The exact solutions of the heat equation are energy stable i.e.

$$E(t) \leq E(0) \quad (\forall t \in (0, T))$$

with
$$E(t) = \frac{1}{2} \int_0^1 u^2(x, t) dx. \quad \text{--- (4)}$$

We will analyze whether the approximate solutions satisfy a discrete version of the energy inequality.

To this end, we define the discrete energy.


$$E_j^n = \frac{\Delta x}{2} \sum_{j=1}^N (u_j^n)^2 \quad \text{--- (5)}$$

We also need several elementary identities: (easy to check):

① Discrete chain rule:

use



$$\omega^n D_t^+ \omega_n = \frac{1}{2} D_t^+ (\omega^n)^2 - \frac{\Delta t}{2} (D_t^+ \omega^n)^2 \quad (6)$$

Proof:

$$\begin{aligned} \omega_n D_t^+ \omega_n &= \frac{\omega_n}{\Delta t} (\omega_{n+1} - \omega_n) \\ &= \frac{1}{2\Delta t} (\omega_{n+1}^2 - \omega_n^2 - (\omega_{n+1} - \omega_n)^2) \\ &= \frac{1}{2} D_t^+ (\omega^n)^2 - \frac{\Delta t}{2} (D_t^+ \omega^n)^2 \end{aligned}$$

② Summation by Parts

$$\begin{aligned} \sum_{j=1}^N \omega_j D_x^- D_x^+ \omega_j &= - \sum_{j=0}^N (D_x^+ \omega_j)^2 \\ &\quad + \omega_{N+1} D_x^+ \omega_N \\ &\quad - \omega_0 D_x^+ \omega_0 \quad (7) \end{aligned}$$

Now as in the continuous case; we multiply both sides of the finite difference scheme (3) by u_j^n .

$$u_j^n D_t^+ u_j^n = u_j^n D_x^- D_x^+ u_j^n$$

by discrete chain rule (6),

$$\frac{1}{2} D_t^+ (u_j^n)^2 = \frac{\Delta t}{2} (D_t^+ u_j^n)^2 + u_j^n D_x^- D_x^+ u_j^n$$

Now using the scheme (3) we obtain;

$$\begin{aligned} \frac{1}{2} D_t^+ (u_j^n)^2 &= \frac{\Delta t}{2} (D_x^- D_x^+ u_j^n)^2 + u_j^n D_x^- D_x^+ u_j^n \\ &= \frac{\Delta t}{2\Delta x^2} ((D_x^+ u_j^n - D_x^- u_j^n)^2) + u_j^n D_x^- D_x^+ u_j^n \end{aligned}$$

In the above we use the identity;

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$$D_x^- D_x^+ u_j^n = \frac{1}{\Delta x} (D_x^+ u_j^n - D_x^- u_j^n)$$

Now we sum the above over j and multiply both sides by $\Delta x \Delta t$

~~$$\frac{1}{2} \sum_j D_t^+ (u_j^n)^2 = \frac{\Delta t}{2\Delta x^2}$$~~

$$\frac{\Delta x \Delta t}{2} \sum_j (D_t^+ (u_j^n)^2) = \frac{\Delta t^2}{2\Delta x} \sum_j (D_x^+ u_j^n - D_x^- u_j^n)^2$$

$$\frac{\Delta x \Delta t}{2} \sum_j u_j^n D_x^- D_x^+ u_j^n$$

We deal with each term separately;

$$T_1 = \frac{\Delta x \Delta t}{2} \sum_{j=1}^N D_t^+ (u_j^n)^2 = \frac{\Delta x}{2} \sum_{j=1}^N (u_{j+1}^n)^2 - \frac{\Delta x}{2} \sum_{j=1}^N (u_j^n)^2$$

$$= E_j^{n+1} - E_j^n$$

$$T_3 := \Delta x \Delta t \sum_{j=1}^N u_j^n D_x^- D_x^+ u_j^n = -\Delta x \Delta t \sum_{j=0}^N (D_x^+ u_j^n)^2$$

(by summation by parts (7))

~~$$T_2 := \frac{\Delta t^2}{2\Delta x} \sum_j (D_x^+ u_j^n - D_x^- u_j^n)^2$$~~
~~$$\leq \frac{\Delta t^2}{\Delta x} \sum_j (D_x^+ u_j^n)^2 + \frac{\Delta t^2}{\Delta x} \sum_j (D_x^- u_j^n)^2$$~~

(using $(a-b)^2 \leq 2(a^2+b^2)$)

$$E_3 = \frac{\Delta t^2}{2\Delta x} \sum_{i=1}^N (D_x^+ u_j^n - D_x^- u_j^n)^2$$

$$\leq \frac{\Delta t^2}{\Delta x} \sum_{j=0}^N (D_x^+ u_j^n)^2 + \frac{\Delta t^2}{\Delta x} \sum_{j=1}^{N+1} (D_x^- u_j^n)^2$$

$$((a-b)^2 \leq 2(a^2 + b^2))$$

Observe that $\sum_{j=0}^N (D_x^+ u_j^n)^2 = \sum_{j=1}^{N+1} (D_x^- u_j^n)^2$ (change indices)

$$E_3 \leq \frac{2\Delta t^2}{\Delta x} \sum_{j=0}^N (D_x^+ u_j^n)^2$$

Hence, combining the above we have;

$$E^{n+1} = E^n + \left(\frac{2\Delta t^2}{\Delta x} - \Delta x \Delta t \right) \sum_{j=0}^N (D_x^+ u_j^n)^2 \quad \text{--- (8)}$$

Note if we assume;

$$\frac{2\Delta t^2}{\Delta x} - \Delta x \Delta t \leq 0$$

$$\Rightarrow \frac{\Delta t}{\Delta x^2} \leq \frac{1}{2} \text{ or } \tau \leq \frac{1}{2}, \quad \text{--- (9)}$$

then $E^{n+1} < E^n \dots < E^0$

or in other words, the energy of the system decreases over time.

if $\tau > \frac{1}{2}$, then we also observe that energy is added to the system (see (8)) and the system is unstable !!!

The condition (9) is termed as the

Courant-Friedrichs-Lewy (CFL) condition.

and the scheme (3) is termed conditionally stable.

Discrete Maximum principle: A different method of obtaining the

CFL condition is from the discrete maximum principle.

Note that (2) can be written as,

$$v_j^{n+1} = \lambda v_{j+1}^n + (1-2\lambda) v_j^n + \lambda v_{j-1}^n, \quad \forall 1 \leq j \leq N; \quad \forall n \quad (10)$$

if we assume that

$$\lambda = \frac{\Delta t}{\Delta x^2} \leq \frac{1}{2}, \quad \text{then}$$

$$2\lambda \leq 1$$

Let $\bar{v}_j^n = \max(v_{j-1}^n, v_j^n, v_{j+1}^n)$

clearly, $v_{j+1}^n \leq \bar{v}_j^n, v_j^n \leq \bar{v}_j^n, v_{j-1}^n \leq \bar{v}_j^n$

Hence, as $\lambda \geq 0, 1-2\lambda \geq 0$, we have from (10) that

$$\begin{aligned} v_j^{n+1} &\leq \lambda \bar{v}_j^n + (1-2\lambda) \bar{v}_j^n + \lambda \bar{v}_{j+1}^n \\ &\leq \bar{v}_j^n \end{aligned}$$

or $v_j^{n+1} \leq \max(v_{j-1}^n, v_j^n, v_{j+1}^n)$

Similarly if $\underline{v}_j^n = \min(v_{j-1}^n, v_j^n, v_{j+1}^n)$

then we have from (10) that

$$v_j^{n+1} \geq \underline{v}_j^n = \min(v_{j-1}^n, v_j^n, v_{j+1}^n)$$

Thus, under the CFL condition (9), the solutions of (3) obey a discrete maximum principle;

$$\min(u_{j-1}^n, u_j^n, u_{j+1}^n) \leq u_j^{n+1} \leq \max(u_{j-1}^n, u_j^n, u_{j+1}^n)$$

$$\forall n = 0, \dots, M$$
$$j = 1, \dots, N$$

Hence iterating over n and j , we obtain:

~~$$u_j^{n+1} \leq \max(0, \max_j u_j^0)$$~~
$$\min(0, \min_j u_j^0) \leq u_j^{n+1} \leq \max(0, \max_j u_j^0), \quad \forall n.$$

This is a discrete version of the maximum principle.

Truncation error: let $u_j^n = u(x_j, t^n)$ with u being the exact solution of the heat equation (1). Then the truncation error is defined as:

$$\tau_j^n = D_t^+ u_j^n - D_x^- D_x^+ u_j^n$$

We can show that $\forall n, j$

$$|\tau_j^n| \leq C(\Delta t + \Delta x^2).$$

By combining the above bound on the truncation error with energy stability, one can prove the following convergence rate: $\forall n$

$$\frac{\Delta x}{2} \sum_{j=1}^N (u_j^n \cdot v_j^n)^2 \leq \bar{C}(\Delta t + \Delta x^2)$$

Thus the rate of convergence of the finite difference scheme is 1 in time and 2 in space.

An Implicit finite difference scheme.

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The CFL condition (9) is quite constraining as $\Delta t \sim \Delta x^2$ is soooo, resulting in very small time steps. To remedy this, we use an implicit finite difference scheme:

$$D_t^- u_j^{n+1} = D_x^+ D_x^+ u_j^{n+1} \quad \text{--- (11)}$$

$$\text{or} \quad \frac{u_j^{n+1} - u_j^n}{\Delta t} = \frac{u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}}{\Delta x^2}, \quad \forall 0 \leq n \leq M, 1 \leq j \leq N$$

$$\text{or:} \quad -\lambda u_{j-1}^{n+1} + (1+2\lambda) u_j^{n+1} - \lambda u_{j+1}^{n+1} = u_j^n \quad \forall 1 \leq j \leq N$$

at any given time level, the scheme (11) reduces to a Matrix Equation:

$$AV^{n+1} = F^n \quad \text{--- (12)}$$

with $V^{n+1} = \{u_j^{n+1}\}_{j=1}^N$ being the vector of unknowns,

$F^n = \{u_j^n\}_{j=1}^N$ being the RHS and

$$A = \begin{bmatrix} 1+2\lambda & -\lambda & 0 & \dots & 0 \\ -\lambda & 1+2\lambda & -\lambda & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & -\lambda & 1+2\lambda \end{bmatrix}$$

being the matrix.

The scheme (11) is implicit as it involves a backward Euler time stepping.

Energy stability:

As for the explicit scheme, we perform an energy stability analysis for the implicit scheme (11);

To this end, we need the following chain rule;

$$w^{n+1} D_t^- w^{n+1} = \frac{1}{2} D_t^- (w_{n+1}^2) + \frac{\Delta t}{2} (D_t^- w^{n+1})^2$$

Then, multiply both sides of eq (11) by u_j^{n+1} and sum over j , also multiply $\Delta x \Delta t$ to both sides;

~~$$\Delta x \Delta t$$~~

$$\begin{aligned} \Delta x \Delta t \sum_{j=1}^N u_j^{n+1} D_t^- u_j^{n+1} &= \Delta x \Delta t \sum_{j=1}^N u_j^{n+1} D_x^- D_x^+ u_j^{n+1} \\ &= \frac{\Delta x \Delta t}{2} \sum_{j=1}^N D_t^- ((u_j^{n+1})^2) = - \Delta x \Delta t \sum_{j=0}^N (D_x^+ u_j^{n+1})^2. \end{aligned}$$

~~$$= \frac{\Delta x \Delta t}{2} \sum_{j=1}^N D_t^- ((u_j^{n+1})^2)$$~~

$$= - \frac{\Delta x \Delta t^2}{2} \sum_{j=0}^N (D_t^- u_j^{n+1})^2$$

$$\Rightarrow E^{n+1} = E^n - \Delta x \Delta t \sum_{j=0}^N (D_x^+ u_j^{n+1})^2 - \frac{\Delta x \Delta t^2}{2} \sum_{j=0}^N (D_t^- u_j^{n+1})^2$$

$$\text{or } E^{n+1} \leq E^n, \quad \forall n.$$

Thus the energy is decreasing every time-step. Thus the

Implicit Scheme (11) is unconditionally stable.

Discrete Maximum principle:

(118)

The unconditional stability of the ~~see~~ implicit scheme can also be observed from the discrete maximum principle.

Note that the implicit scheme (11) can be written as:

$$(1+2\lambda) v_j^{n+1} = v_j^n + \lambda v_{j+1}^{n+1} + \lambda v_{j-1}^{n+1} \quad (13)$$

Now let; $\bar{v}^{n+1} = \max_{0 \leq j \leq N+1} v_j^{n+1}$

as $\lambda > 0$, (13) implies that:

$$(1+2\lambda) v_j^{n+1} \leq v_j^n + 2\lambda \bar{v}^{n+1}, \quad \forall j$$

~~observe~~ that $\Rightarrow (1+2\lambda) v_j^{n+1} \leq \bar{v}^n + 2\lambda \bar{v}^{n+1}, \quad \forall j, \quad 1 \leq j \leq N$

observe that the rhs is independent of j ,

hence:

$$\max_j (1+2\lambda) v_j^{n+1} \leq \bar{v}^n + 2\lambda \bar{v}^{n+1}$$

$$\Rightarrow (1+2\lambda) \bar{v}^{n+1} \leq \bar{v}^n + 2\lambda \bar{v}^{n+1}$$

$$\Rightarrow \bar{v}^{n+1} \leq \bar{v}^n, \quad \forall n.$$

Similarly we can prove a minimum principle and conclude that:

$$\min(0, \min_{1 \leq j \leq N} u_j^0) \leq v_j^{n+1} \leq \max(0, \max_{1 \leq j \leq N} u_j^0) \quad (14)$$

which is a discrete version of the maximum principle for the heat equation (1).

Again, we do not assume any condition on $\Delta t, \Delta x$ and show that the implicit scheme is unconditionally stable.

We can prove that the convergence rate of the implicit scheme is also 1 in time and 2 in space.

Crank-Nicolson scheme:

Both the explicit and implicit schemes are first-order in time.

To obtain higher-order temporal accuracy, we can use the Crank-Nicolson method i.e.,

$$D_t^+ v_j^n = \frac{1}{2} D_x^- D_x^+ v_j^{n+1} + \frac{1}{2} D_x^- D_x^+ v_j^n \quad (15)$$

Note that (15) is an average of the spatial derivatives at time-steps t^n and t^{n+1} i.e. a linear combination of the ~~Bas~~ Explicit scheme

(3) and the implicit scheme (11) !!!

We have the boundary conditions:

$$v_0^n = v_{N+1}^n = 0, \quad \forall n$$

and the initial conditions:

$$v_j^0 = u_j^0 = u_0(x_j)$$

The scheme can be recast as a Matrix equation:

$$A v^{n+1} = F^n$$

with vectors of unknowns;

$$v^{n+1} = \left\{ v_j^{n+1} \right\}_{j=0}^N$$

Right hand side:

$$f_j^n = \frac{\lambda}{2} v_{j-1}^n + (1-\lambda) v_j^n + \frac{\lambda}{2} v_{j+1}^n$$

By integrating by parts.

(121)

$$E^{n+1} - E^n = -\frac{\Delta x \Delta t}{2} \sum_j (D_x^+ u_j^{n+1})^2 - \frac{\Delta x \Delta t}{2} \sum_j D_x^+ u_j^n D_x^+ u_j^{n+1} - \frac{\Delta x \Delta t}{2} \sum_j (D_x^+ u_j^n)^2$$

$$\Rightarrow E^{n+1} = E^n - \frac{\Delta x \Delta t}{2} \sum_j (D_x^+ u_j^{n+1} + D_x^+ u_j^n)^2$$

$$\Rightarrow E^{n+1} < E^n, \quad \forall n$$

Thus the Crank-Nicolson scheme is unconditionally energy stable.

Truncation errors: Let u be the exact solution of the Heat equation (1)

$$\text{Define } u_j^n = u(x_j, t^n)$$

Then the truncation error is defined by.

$$c_j^n := D_t^+ u_j^n - \frac{1}{2} D_x^- D_x^+ u_j^{n+1} - \frac{1}{2} D_x^- D_x^+ u_j^n$$

By Taylor expansion, one can show that

$$|c_j^n| \leq C(\Delta t^2 + \Delta x^2)$$

Using the above truncation error and the energy estimate, one can prove a second-order convergence of the Crank-Nicolson scheme in both space and time.