Prüfung Numerische Mathematik

Fr 08

(1)

x_i	1	2	3
f_i	1	3	4

a) mit linearer interpolation.

$$f(x) \approx \frac{x_R - x}{h} f(x_L) + \frac{x - x_L}{h} f(x_R) \quad (1)$$

$$\int_1^3 f(x)\,dx = \int_1^2 f(x)\,dx + \int_2^3 f(x)\,dx \quad (1)$$

für $1 \le x \le 2$ $\quad f(x) \approx (2-x)\cdot 1 + (x-1)\cdot 3 = 2x-1$

für $2 \le x \le 3$ $\quad f(x) \approx (3-x)\cdot 3 + (x-2)\cdot 4 = (x+1) \quad (1)$

$$\int_1^3 f(x)\,dx \approx \int_1^2 2x-1\,dx + \int_2^3 x+1\,dx$$

$$= \left[\frac{2x^2}{2} - x\right]_1^2 + \left[\frac{x^2}{2} + x\right]_2^3$$

$$= 5.5. \quad (1)$$

b) mit Lagrange interpolation

$$f(x) \approx \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} f(x_0) + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} f(x_1) + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} f(x_2) \quad (2)$$

$$= \frac{(x-2)(x-3)}{(-1)(-2)}\cdot 1 + \frac{(x-1)(x-3)}{(1)(-1)}\cdot 3 + \frac{(x-1)(x-2)}{(2)(1)}\cdot 4$$

$$= \frac{1}{2}(x^2-5x+6) - 3(x^2-4x+3) + 2(x^2-3x+2) \quad (1)$$

$$\int_1^3 f(x)\,dx \approx \left[\frac{1}{2}\left(\frac{x^3}{3} - \frac{5x^2}{2} + 6x\right) - 3\left(\frac{x^3}{3} - \frac{4x^2}{2} + 3x\right) + 2\left(\frac{x^3}{3} - \frac{3x^2}{2} + 2x\right)\right]_1^3$$

$$= \frac{1}{3} + \frac{4}{3}(3) + 4\cdot\frac{1}{3} = \frac{17}{3} \approx 5.6667 \quad (1)$$

(2)

i) C.G convergent für positiv definit symmetrisch Matrizen.

ii) Matrix $A^{(I)}$: ist symmetrisch.

$$Eig(A-\lambda I) \Rightarrow (2-\lambda)\begin{vmatrix} 2-\lambda & 0 \\ 0 & 2-\lambda \end{vmatrix} + 1\begin{vmatrix} -1 & 0 \\ -10 & 2-\lambda \end{vmatrix} - 10\begin{vmatrix} -1 & 2-\lambda \\ -10 & 0 \end{vmatrix}$$

$$= (2-\lambda)^3 - (2-\lambda) - 10(10(2-\lambda)) = 0$$

$$(2-\lambda)(-1-100+(2-\lambda)^2) = 0$$

$\lambda = 2$ ist ein Lösung

$$-101 + 4 - 4\lambda + \lambda^2 = 0$$

$$-97 - 4\lambda + \lambda^2 = 0 \qquad \lambda = \frac{4 \pm \sqrt{16+4\times 97}}{2}$$

(2)

$$= 12.049, -8.0498$$

Nicht positive definit.

Matrix $A^{(II)}$: ist symmetrisch

$$Eig(A-\lambda I) \Rightarrow (2-\lambda)\begin{vmatrix} (2-\lambda) & 0 \\ 0 & (2-\lambda) \end{vmatrix} + 1\begin{vmatrix} -1 & 0 \\ 0 & 2-\lambda \end{vmatrix} + 0\begin{vmatrix} -1 & 2-\lambda \\ 0 & 0 \end{vmatrix}$$

$$(2-\lambda)^3 - (2-\lambda) = 0$$

$\lambda = 2$ ist ein Lösung

$$(2-\lambda)^2 - 1 = 0$$

$$3 - 4\lambda + \lambda^2 = 0 \qquad \lambda = \frac{4 \pm \sqrt{16-12}}{2} = 3, 1$$

(2)

ist positive definit.

Nur für $A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ ist C.G. convergiert.

(1)

b) 1 schritt.

Algothramus in Lecture ist für $Au + d = 0$. Sei $d = -b$.

(2)

$d = -b = -(1,1,1)^T$

$$v_0 = (1,1,1)^T \qquad r_0 = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \qquad p_1 = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$$

$$\rho_1 = \frac{(r_0, r_0)}{(p_1, Ap_1)} = \frac{1}{2} \qquad u_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1/2 \end{pmatrix}$$

$$r_1 = r_0 + \rho_1 Ap_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 \\ 0 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

c) C.G ist exakt nach n Schritten. Also ...

(1)

3) Cubic Spline

x_i	0	1	2	3
f_i	0	2	1	-1

Interior gleichungen $(n-2 = 2)$

$$b_1 f_1' + a_1 f_2' + b_2 f_3' = d_2$$
$$b_2 f_2' + a_2 f_3' + b_3 f_4' = d_3 \quad 1$$

$$\begin{pmatrix} b_1 & a_1 & b_2 & 0 \\ 0 & b_2 & a_2 & b_3 \end{pmatrix} \begin{pmatrix} f_1' \\ f_2' \\ f_3' \\ f_4' \end{pmatrix} = \begin{pmatrix} d_2 \\ d_3 \end{pmatrix} \quad 1$$

mit $f_1' = 0$

$$\begin{pmatrix} a_1 & b_2 & 0 \\ b_2 & a_2 & b_3 \end{pmatrix} \begin{pmatrix} f_2' \\ f_3' \\ f_4' \end{pmatrix} = \begin{pmatrix} d_2 \\ d_3 \end{pmatrix} \quad 1$$

mit "Not a Knot"

$$\begin{pmatrix} a & b_2 & 0 \\ b_2 & a_2 & b_3 \\ 0 & \frac{1}{h_3}+\frac{1}{h_2} & \frac{1}{h_3} \end{pmatrix} \begin{pmatrix} f_2' \\ f_3' \\ f_4' \end{pmatrix} = \begin{pmatrix} d_2 \\ d_3 \\ 2c_3 + \frac{h_3}{h_3+h_2}(c_3+c_2) \end{pmatrix} \quad 1$$

$h_1 = h_2 = h_3 = 1$

$a_i = \frac{2}{h_i} + \frac{2}{h_{i+1}} = 4 \Rightarrow a_1 = a_2 = a_3 = 4$

$b_i = \frac{1}{h_i} = 1 \Rightarrow b_1 = b_2 = b_3 = 1$

$c_i = \frac{f_{i+1} - f_i}{h_i^2} \qquad c_1 = \frac{2-0}{1} = 2, \quad c_2 = \frac{1-2}{1} = -1, \quad c_3 = \frac{-1-1}{1} = -2$

$d_i = 3(c_i + c_{i+1}) \qquad d_2 = 3(2-1) = 3 \qquad d_3 = 3(-1-2) = -9 \quad 1$

$$\begin{pmatrix} 4 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} f_2' \\ f_3' \\ f_4' \end{pmatrix} = \begin{pmatrix} 3 \\ -9 \\ -4 + \frac{1}{2}(-3) \end{pmatrix} = \begin{pmatrix} 3 \\ -9 \\ -5.5 \end{pmatrix} \quad 1 \quad \leadsto \begin{pmatrix} f_2' \\ f_3' \\ f_4' \end{pmatrix} = \begin{pmatrix} 1.3571 \\ -2.4286 \\ -0.6429 \end{pmatrix}$$

$a_0 = 2$, $a_1 = 1$ $\rightarrow$ $b_0 = -1$, $b_{-1} = 1.3571$, $b_1 = -2.4286$ $\rightarrow$ $c_0 = -2.3571$, $c_1 = -1.4286$ $\rightarrow$ $d_0 = 0.9283$. Schema von Skript.

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$Q(t) = a_0 + (b_0 + (c_0 + d_0 t)(t-1))t$

$= 2 + [-1 + (-2.3571 + 0.9285t)(t-1)]$

$t = \frac{1.5-1}{1} = \frac{1}{2}$

$Q(\frac{1}{2}) = 1.973125. \quad 1$

③ $3x_1 + \sin 2x_2 = 0$

$2x_1 x_2 - 3 = 0$

Matlab converges to

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0.28679 \\ 5.2304 \end{pmatrix}$$

initial condition $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ \pi/2 \end{pmatrix}$ ①

$$J = \begin{pmatrix} 3 & 2\cos 2x_2 \\ 2x_2 & 2x_1 \end{pmatrix} \quad ① \qquad J\Big|_{\substack{x_1 = 2 \\ x_2 = \frac{\pi}{2}}} = \begin{pmatrix} 3 & -2 \\ \pi & 4 \end{pmatrix} \quad ①$$

$$LR = \underbrace{\begin{pmatrix} 1 & 0 \\ \frac{\pi}{3} & 1 \end{pmatrix}}_{L} \underbrace{\begin{pmatrix} 3 & -2 \\ 0 & \frac{2\pi+12}{3} \end{pmatrix}}_{R} = \begin{pmatrix} 3 & -2 \\ \pi & 4 \end{pmatrix} \quad ①$$

Step 1

$$f(x_0) = \begin{pmatrix} 6 \\ \frac{4\pi}{2} - 3 \end{pmatrix}$$

$LR\Delta = -f(x_0)$

Solve $Ly = -f(x_0)$

then $R\Delta = y$

$$\begin{pmatrix} 1 & 0 \\ \frac{\pi}{3} & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} -6 \\ 3 - 2\pi \end{pmatrix} \Rightarrow \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} -6 \\ +3 \end{pmatrix}$$

$$\begin{pmatrix} 3 & -2 \\ 0 & \frac{2\pi+12}{3} \end{pmatrix} \begin{pmatrix} \Delta_1 \\ \Delta_2 \end{pmatrix} = \begin{pmatrix} -6 \\ 3 \end{pmatrix} \Rightarrow \begin{pmatrix} \Delta_1 \\ \Delta_2 \end{pmatrix} = \frac{1}{2\pi+12} \begin{pmatrix} -18 - 4\pi \\ 9 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ \pi/2 \end{pmatrix} + \frac{1}{2\pi+12} \begin{pmatrix} -18 - 4\pi \\ 9 \end{pmatrix} \approx \begin{pmatrix} 0.3282 \\ 2.0631 \end{pmatrix} \quad ②$$

Step 2

$$f(x_1) = \begin{pmatrix} 0.151543 \\ -1.64578 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 \\ \frac{\pi}{3} & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} -0.151545 \\ 1.64578 \end{pmatrix} \Rightarrow \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} -0.151543 \\ 1.80447 \end{pmatrix}$$

$$\begin{pmatrix} 3 & -2 \\ 0 & \frac{2\pi+12}{3} \end{pmatrix} \begin{pmatrix} \Delta_1 \\ \Delta_2 \end{pmatrix} = \begin{pmatrix} -0.151545 \\ 1.80447 \end{pmatrix} \Rightarrow \begin{pmatrix} \Delta_1 \\ \Delta_2 \end{pmatrix} = \begin{pmatrix} 0.1469086 \\ 0.296088 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0.3282 \\ 2.0631 \end{pmatrix} + \begin{pmatrix} 0.1469086 \\ 0.296088 \end{pmatrix} = \begin{pmatrix} 0.475 \\ 2.3592 \end{pmatrix} \quad ②$$

⑤

$$A = \begin{pmatrix} \frac{4}{3\sqrt{2}} & \frac{4}{3\sqrt{2}} \\ -\frac{4}{3\sqrt{2}} & -\frac{4}{3\sqrt{2}} \\ \frac{2}{3\sqrt{2}} & \frac{2}{3\sqrt{2}} \end{pmatrix} \qquad c = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \qquad U = \begin{pmatrix} 2/3 & 1/3 & 2/3 \\ -2/3 & 2/3 & 1/3 \\ 1/3 & 2/3 & -2/3 \end{pmatrix}$$

$$V = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

$A = USV^T$

U, V are orthogonal matrices $U^{-1} = U^T$ and $V^{-1} = V^T$ ②

$U^T A = SV^T$ and $U^T A V = S$

$$S = \begin{pmatrix} 2/3 & -2/3 & 1/3 \\ 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \end{pmatrix} \begin{pmatrix} \frac{4}{3\sqrt{2}} & \frac{4}{3\sqrt{2}} \\ \frac{-4}{3\sqrt{2}} & \frac{-4}{3\sqrt{2}} \\ \frac{2}{3\sqrt{2}} & \frac{2}{3\sqrt{2}} \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

②

Therefore the Matrix A is of rank 1. $(= K)$

In general $x = Vy = \sum_{i=1}^{K} y_i V^{(i)} + \sum_{K+1}^{n} y_i V^{(i)}$

②

with $y_i = \frac{1}{s_i} (U^{(i)})^T c$.

For $i > 1$ the values of y_i are infinite. To obtain the solution with the smallest 2-Norm we take

$$x^* = \sum_{i=1}^{(K=1)} y_i V^{(i)} \qquad y_1 = \frac{1}{s_1} U^{(1)T} c = \frac{1}{2} \begin{pmatrix} 2/3 & -2/3 & 1/3 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \frac{1}{6}$$

$$= \frac{1}{6} \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} = \frac{1}{6\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

②

6) Euler explicit

$$\tilde{x}_{n+1} = \tilde{x}_n + h f(t_n, \tilde{x}_n)$$

Trapozidal rule

$$\hat{x}_{n+1} = \hat{x}_n + \frac{h}{2}\left(f(t_n, \hat{x}_n) + f(t_{n+1}, \hat{x}_{n+1})\right)$$

use Euler explicit as a predicator

The P-C scheme is

$$x^*_{n+1} = x_n + f(t_n, x_n)h$$

$$x_{n+1} = x_n + \frac{h}{2}\left(f(t_n, x_n) + f(t_{n+1}, x^*_{n+1})\right) \quad ②$$

a) Apply to $\dot{x} = \exp(xt/2)$ with $x(0) = 1$, $h = 0.1$

$f(t, x) = \exp(xt/2)$ ①

Step 1

$$x_1^* = x_0 + hf(t_0, x_0) = 1 + 0.1(\exp(1 \times 0/2)) = 1.1$$

$$x_1 = x_0 + \frac{h}{2}\left(f(t_0, x_0) + f(t_{n+1}, x^*_{n+1})\right) = 1 + \frac{0.1}{2}\left(e^{\frac{1 \times 0}{2}} + e^{\frac{1.1 \times 0.1}{2}}\right)$$

$$= 1 + 0.05(1 + 1.056)$$

$$= 1.102827 \quad ①$$

Step 2

$$x_2^* = 1.102827 + 0.1(\exp(1.102827 \times 0.1/2)) = 1.208496$$

$$x_2 = 1.102827 + 0.05\left(\exp\left(\frac{1.102827 \times 0.1}{2}\right) + \exp\left(\frac{1.208496 \times 0.2}{2}\right)\right)$$

$$= 1.2120842 \quad ①$$

b) $x_{n+1} = x_n + \frac{h}{2}\left(f(t_n, x_n) + f(t_{n+1}, x_n + hf(t_n, x_n))\right)$

$x_{n+1} = x_n + h\sum_{i=1}^{2} b_i K_i$ $b_1 = 1/2$ $K_1 = f(t_n, x_n)$

$b_2 = 1/2$ $K_2 = f(t_{n+1}, x_n + hf(t_n, x_n))$

$K_i = f(t_n + c_i h, x + h\sum_{j=1}^{i-1} a_{ij} K_j)$ $c_1 = 0$ $a_{11} = 0$

$c_2 = 1$ $a_{21} = 1$ $a_{22} = 0$

$$\begin{array}{c|cc} 0 & 0 & \\ 1 & 1 & 0 \\ \hline & 1/2 & 1/2 \end{array} \quad ②$$

c) explicit ①