Numerische Mathematik

D-MAVT

Frühling 2006.

Musterlösung.

① $A = L + R + D \Rightarrow L = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} \quad D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \quad R = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}$

For the Jacobi Method.

$$T_J = -D^{-1}(L+R) \qquad c_J = D^{-1} b.$$

$$= \begin{pmatrix} -1/2 & 0 & 0 \\ 0 & -1/2 & 0 \\ 0 & 0 & -1/2 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1/2 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1/2 & 0 \end{pmatrix}$$

$$c_J = \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/2 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1 \\ 3/2 \end{pmatrix}$$

Gauss-Seidal

$$T_{GS} = -(D+L)^{-1} R = -\begin{pmatrix} 2 & 0 & 0 \\ -1 & 2 & 0 \\ 0 & -1 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 2 & 0 & 0 & 0 & -1 & 0 \\ -1 & 2 & 0 & 0 & 0 & -1 \\ 0 & -1 & 2 & 0 & 0 & 0 \end{array}\right) \quad \left(\begin{array}{ccc|ccc} 2 & 0 & 0 & 0 & -1 & 0 \\ 0 & 2 & 0 & 0 & -1/2 & -1 \\ 0 & -1 & 2 & 0 & 0 & 0 \end{array}\right) \quad \left(\begin{array}{ccc|ccc} 2 & 0 & 0 & 0 & -1 & 0 \\ 0 & 2 & 0 & 0 & -1/2 & -1 \\ 0 & 0 & 2 & 0 & -1/4 & -1/2 \end{array}\right)$$

$$T_{GS} = -\begin{pmatrix} 0 & -1/2 & 0 \\ 0 & -1/4 & -1/2 \\ 0 & -1/8 & -1/4 \end{pmatrix} = \begin{pmatrix} 0 & 1/2 & 0 \\ 0 & 1/4 & 1/2 \\ 0 & 1/8 & 1/4 \end{pmatrix}$$

$$c_{GS} = +(D+L)^{-1} b$$

$$\left(\begin{array}{ccc|c} 2 & 0 & 0 & 1 \\ -1 & 2 & 0 & 2 \\ 0 & -1 & 2 & 3 \end{array}\right) \quad \left(\begin{array}{ccc|c} 2 & 0 & 0 & 1 \\ 0 & 2 & 0 & 2.5 \\ 0 & -1 & 2 & 3 \end{array}\right) \quad \left(\begin{array}{ccc|c} 2 & 0 & 0 & 1 \\ 0 & 2 & 0 & 2.5 \\ 0 & 0 & 2 & 4.25 \end{array}\right) \quad c_{GS} = \begin{pmatrix} 0.5 \\ 1.25 \\ 2.125 \end{pmatrix}$$

② ..

T	45	40	37
m	0	5	10

$$T - T_a = \Delta T_0 e^{km}$$

$$\ln(T - T_a) = \ln \Delta T_0 + \ln e^{km} = \ln \Delta T_0 + km$$

$$x = k \,, \quad y = \ln \Delta T_0 \qquad \Rightarrow x + y - \ln(T - T_a) = 0$$

Error equations:

$$\left.\begin{aligned} 0 \cdot x + y - \ln(45-35) &= r_1 \\ 5x + y - \ln(40-35) &= r_2 \\ 10x + y - \ln(37-35) &= r_3 \end{aligned}\right\} \quad \underbrace{\begin{pmatrix} 0 & 1 \\ 5 & 1 \\ 10 & 1 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x \\ y \end{pmatrix}}_{x} - \underbrace{\begin{pmatrix} \ln 10 \\ \ln 5 \\ \ln 2 \end{pmatrix}}_{b} = \underbrace{\begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix}}_{r}$$

Gauss Normal equation Method.

$$A^T A x = A^T b .$$

$$A^T A = \begin{pmatrix} 125 & 15 \\ 15 & 3 \end{pmatrix} \qquad (\text{check } \det(A^T A) = 150 \neq 0)$$

$$A^T b = \begin{pmatrix} 14.9786 \\ 4.6051 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} 125 & 15 & 14.9786 \\ 15 & 3 & 4.6051 \end{array}\right) \qquad \left(\begin{array}{cc|c} 125 & 15 & 14.9786 \\ 0 & 1.2 & 2.80757 \end{array}\right) \qquad \begin{aligned} y &= 2.3397 \\ x &= -0.1609 \end{aligned}$$

$$x = k \approx -0.1609 \qquad \Delta T_0 = e^{y} \approx 10.378$$

$$T - T_a \approx 10.378\, e^{-0.1609\, m}$$

(3)

$$f_1(c,z) = c\left(\cosh\frac{z}{c} - \cosh\left(\frac{z+50}{c}\right)\right) - 10 = 0$$

$$f_2(c,z) = c\left(\sinh\left(\frac{z+50}{c}\right) - \sinh\frac{z}{c}\right) - 65 = 0$$

Note $\frac{d}{dx}(\cosh x) = \sinh x$

$\frac{d}{dx}(\sinh x) = \cosh x$.

- $J = \begin{pmatrix} \partial f_1/\partial c & \partial f_1/\partial z \\ \partial f_2/\partial c & \partial f_2/\partial z \end{pmatrix} = \begin{pmatrix} j_{11} & j_{12} \\ j_{21} & j_{22} \end{pmatrix}$ with enteries.

$$j_{11} = \frac{\partial f_1}{\partial c} = \cosh\frac{z}{c} - \frac{zc}{c^2}\sinh\left(\frac{z}{c}\right) - \cosh\left(\frac{z+50}{c}\right) + \frac{c(z+50)}{c^2}\sinh\left(\frac{z}{c}\right)$$

$$= \cosh\frac{z}{c} - \frac{z}{c}\sinh\left(\frac{z}{c}\right) - \cosh\left(\frac{z+50}{c}\right) + \frac{(z+50)}{c}\sinh\left(\frac{z+50}{c}\right)$$

$$j_{12} = \frac{\partial f_1}{\partial z} = \frac{c}{c}\sinh\frac{z}{c} - \frac{c}{c}\sinh\left(\frac{z+50}{c}\right) = \sinh\frac{z}{c} - \sinh\left(\frac{z+50}{c}\right)$$

- $$j_{21} = \frac{\partial f_2}{\partial c} = \sinh\left(\frac{z+50}{c}\right) - \frac{(z+50)}{c}\cosh\left(\frac{z+50}{c}\right) - \sinh\left(\frac{z}{c}\right) + \frac{z}{c}\cosh\left(\frac{z}{c}\right)$$

$$j_{22} = \frac{\partial f_2}{\partial z} = \cosh\left(\frac{z+50}{c}\right) - \cosh\left(\frac{z}{c}\right)$$

b) $-\lambda\left(\lambda^2 - \frac{1}{4}\right) - \frac{1}{2}\left(\frac{1}{2}\right)(-\lambda) = 0$

$-\lambda^3 + \frac{\lambda}{4} + \frac{\lambda}{4} = 0$

$\lambda\left(-\lambda^2 + \frac{1}{2}\right) = 0 \qquad \lambda = 0, \ \lambda = \pm\frac{1}{\sqrt{2}}.$

$\rho_J = \max|\lambda_i| = \frac{1}{\sqrt{2}} < 1 \quad \therefore$ OK

Note A is not strictly diagonal $|a_{ii}| \ngtr \sum_{j=1, j\neq i}^{3} |a_{ij}|$ so students must check for eigenvalues. This condition does NOT imply that Jacobi does not converge.

ii) $-\lambda\left(\left(\frac{1}{4} - \lambda\right)^2 - \frac{1}{16}\right) - \frac{1}{2}(0) + 0 = 0 \qquad \lambda = 0, 0, \frac{1}{2}$

$\rho_{gs} = \max|\lambda_i| = 1/2 < 1 \quad \therefore$ OK.

Both schemes converge.

Gauss-Siedel: $T_{gs} = \begin{pmatrix} 0 & 1/2 & 0 \\ 0 & 1/4 & 1/2 \\ 0 & 1/8 & 1/4 \end{pmatrix} \quad c_{gs} = \begin{pmatrix} 1/2 \\ 5/4 \\ 17/8 \end{pmatrix}$

$x^{(1)} = T_{gs}\, x^{(0)} + c_{gs}$

$= \begin{pmatrix} 1/2 \\ 3/4 \\ 3/8 \end{pmatrix} + \begin{pmatrix} 1/2 \\ 5/4 \\ 17/8 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 20/8 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 2.5 \end{pmatrix}$

Jacobi

$x^{(1)} = T_J\, x^{(0)} + c_J \qquad T_J = \begin{pmatrix} 0 & 1/2 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1/2 & 0 \end{pmatrix} \quad c_J = \begin{pmatrix} 1/2 \\ 1 \\ 3/2 \end{pmatrix}$

$= \begin{pmatrix} 1/2 \\ 1 \\ 1/2 \end{pmatrix} + \begin{pmatrix} 1/2 \\ 1 \\ 3/2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$

(4)

$$\Phi(z) = \frac{1}{2} + \frac{I(z)}{\sqrt{2\pi}} \quad \text{with} \quad I(z) = \int_0^z \underbrace{e^{-u^2/2}}_{f(u)}\, du$$

We want to compute $\Phi(1.5) = \frac{1}{2} + \frac{I(1.5)}{\sqrt{2\pi}}$.

Trapezeum rule (1 step) $\quad I \approx T_0 = h S_0$

$$S_0 = \frac{1}{2} f(a) + \frac{1}{2} f(b)$$

$h_0 = b - a \Rightarrow b = 1.5,\ a = 0,\ h_0 = 1.5$

$$S_0 = \frac{1}{2} f(0) + \frac{1}{2} f(1.5)$$

$$= \frac{1}{2}(e^0) + \frac{1}{2}\left(e^{-1.5^2/2}\right)$$

$$= 0.662326$$

$T_0 = h_0 S_0 = 1.5 \times 0.662326 = 0.993489 \quad (= I_0)$

$$\Phi(1.5) \approx \frac{1}{2} + \frac{0.993489}{\sqrt{2\pi}} = 0.896344.$$

2 steps

$h_1 = \frac{h_0}{2} = \frac{3}{4}. \qquad S_1 = S_0 + f(a + h_1)$

$$= 0.662326 + e^{-(3/4)^2/2}$$

$$= 1.417165$$

$T_1 = S_1 h_1 = 1.417165 \times \frac{3}{4} = 1.0628742 \quad (= I_1)$

$$\Phi(1.5) \approx \frac{1}{2} + \frac{1.0628742}{\sqrt{2\pi}} = 0.924025.$$

Romburg

$I_0 = I + c_0 h_0^2 + O(h_0^2) \qquad I_1 = I + c_0 h_1^2 + O(h_1^2)$

$$= I + c_0\left(\frac{h_0}{2}\right)^2 + O(h_0^2)$$

$\Rightarrow$

$I \approx \frac{4I_1 - I_0}{3} = 1.0860026$

$$\Phi(1.5) \approx \frac{1}{2} + \frac{1.0860026}{\sqrt{2\pi}} = 0.93325.$$

(5)

x_i	1	2	3	4
f_i'	2	1	3	2

Determine f_i's.

Period $= 4$.

Interior equations $(n-2)$ $(=2)$

$$\begin{pmatrix} b_1 & a_1 & b_2 & 0 \\ 0 & b_2 & a_2 & b_3 \end{pmatrix} \begin{pmatrix} f_1' \\ f_2' \\ f_3' \\ f_4' \end{pmatrix} = \begin{pmatrix} d_2 \\ d_3 \end{pmatrix}$$

BC. $8f_1' + 2f_2' + 2f_3' = \frac{6}{h} f_2 - \frac{2}{h} f_3 \qquad f_1' = f_4'$

$$\begin{pmatrix} b_1 & a_1 & b_2 \\ b_3 & b_2 & a_2 \\ 8 & 2 & 2 \end{pmatrix} \begin{pmatrix} f_1' \\ f_2' \\ f_3' \end{pmatrix} = \begin{pmatrix} d_2 \\ d_3 \\ \frac{1}{h}(6f_2 - 2f_3) \end{pmatrix}$$

$h_1 = 1 \quad (= h_2 - h_3)$

$a_1 = 4 \quad a_2 = 4$

$b_1 = 1 \quad b_2 = 1 \quad b_3 = 1$

$d_2 = 3(c_1 + c_2) = 3/h^2 (f_2 - f_1 + f_3 - f_2) = 3/h^2 (f_3 - f_1)$

$d_3 = 3(c_2 + c_3) = 3/h^2 (f_3 - f_2 + f_4 - f_3) = 3/h^2 (f_4 - f_2)$

$$\begin{pmatrix} b_1 & a_1 & b_2 \\ b_3 & b_2 & a_2 \\ 8 & 2 & 2 \end{pmatrix} \begin{pmatrix} f_1' \\ f_2' \\ f_3' \end{pmatrix} = \begin{pmatrix} 1 & 4 & 1 \\ 1 & 1 & 4 \\ 8 & 2 & 2 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 9 \\ 15 \\ 24 \end{pmatrix} = \begin{pmatrix} -3/h^2 & 0 & 3/h^2 \\ 3/h^2 & -3/h^2 & 0 \\ 0 & 6/h & -2/h \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} -3 & 0 & 3 & 9 \\ 3 & -3 & 0 & 15 \\ 0 & 6 & -2 & 24 \end{array}\right) \quad \left(\begin{array}{ccc|c} -3 & 0 & 3 & 9 \\ 0 & -3 & 3 & 24 \\ 0 & 6 & -2 & 24 \end{array}\right) \quad \left(\begin{array}{ccc|c} -3 & 0 & 3 & 9 \\ 0 & -3 & 3 & 24 \\ 0 & 0 & 4 & 72 \end{array}\right)$$

$f_3 = 18$

$f_2 = 10$

$f_1 = f_4 = 15$

6 i)

$$\begin{array}{c|ccc} 0 & 0 & & \\ 1/2 & 1/2 & 0 & \\ 1 & -1 & 2 & 0 \\ \hline & 1/6 & 4/6 & 1/6 \end{array} \qquad \begin{array}{c|cc} 0 & 0 & \\ 1 & 1/2 & 1/2 \\ \hline & 1/2 & 1/2 \end{array}$$

$K_1 = f(t_n, x_n)$

$K_2 = f(t_n + \frac{1}{2}h, x_n + \frac{1}{2}K_1 h)$ $\qquad \dot{x} = f(t, x)$

$K_3 = f(t_n + h, x_n - hK_1 + 2K_2 h)$

$$x_{n+1} = x_n + \frac{h}{6}(K_1 + 4K_2 + K_3)$$

explicit. No dependence of x_{n+1} in K_1, K_2, or K_3.

ii) $K_1 = f(t_n, x_n)$

$K_2 = f(t_n + h, x_n + \frac{1}{2}h(K_1 + K_2)) = f(t_n + h, x_{n+1})$

$$x_{n+1} = x_n + \frac{h}{2}(K_1 + K_2)$$

b) i) $x_{n+1} = x_n + \frac{h}{6}\left(\lambda x_n + 4\lambda(x_n + \frac{1}{2}h x_n) + \lambda(x_n - h x_n + 2\lambda(x_n + \frac{1}{2}hx\right.$

$$= x_n + h\lambda x_n + \frac{2\lambda^2 x_n h^2}{6} - \frac{h^2\lambda^2 x_n}{6} + \frac{2\lambda^2 h^2 x_n}{6} + \frac{\lambda^3 h^3 x_n}{6}$$

$$= (1 + h\lambda + \frac{1}{2}h^2\lambda^2 + \frac{1}{6}h^3\lambda^3)\, x_n$$

$$R(h\lambda) = \left|\frac{x_{n+1}}{x_n}\right| = \left|1 + h\lambda + \frac{1}{2}h^2\lambda^2 + \frac{1}{6}h^3\lambda^3\right|$$

ii) $x_{n+1} = x_n + \frac{h}{2}(\lambda x_n + \lambda x_{n+1})$

$$\left(1 - \frac{h\lambda}{2}\right) x_{n+1} = \left(1 + \frac{1}{2}h\lambda\right) x_n$$

$$R(h\lambda) = \left|\frac{x_{n+1}}{x_n}\right| = \left|\frac{1 + \frac{1}{2}h\lambda}{1 - \frac{1}{2}h\lambda}\right| = \left|\frac{2 + h\lambda}{2 - h\lambda}\right|$$

(c) $\dot{x} = Ax$ with $A = \begin{pmatrix} 1 & 3 \\ 0 & 2 \end{pmatrix}$

Eigen values $\lambda = 1$, $\lambda = 2$

$\lambda = 1$ $\left.\begin{matrix} 0 & 3 \\ 0 & 1 \end{matrix}\right| \begin{matrix} 0 \\ 0 \end{matrix}$ $\quad v^{(1)} = \text{span} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\lambda = 2$ $\left.\begin{matrix} -1 & 3 \\ 0 & 0 \end{matrix}\right|$ $\quad v^{(2)} = \text{span} \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

Exact solution for $x(0) = v^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$x(t) = c_1 e^{\lambda_1 t} v_{(1)} + c_2 e^{\lambda_2 t} v_{(2)}$$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = c_1 e^{t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} 3 \\ 1 \end{pmatrix} \Big|_{t=0} \Rightarrow c_1 = 1, \; c_2 = 0$$

$$x(t) = e^{t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad x(t=0.1) = e^{0.1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1.10517 \\ 0 \end{pmatrix}$$

$h = 0.1$ Explicit scheme

$$K_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad K_2 = \begin{pmatrix} 1 & 3 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 + 1/2(1 \times h) \\ 0 + 0 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1.05 \\ 0 \end{pmatrix} = \begin{pmatrix} 1.05 \\ 0 \end{pmatrix}$$

$$K_3 = \begin{pmatrix} 1 & 3 \\ 0 & 2 \end{pmatrix} \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} - h \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 2 \begin{pmatrix} 1.05 \\ 0 \end{pmatrix} h \right) = \begin{pmatrix} 1 & 3 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1.11 \\ 0 \end{pmatrix} = \begin{pmatrix} 1.11 \\ 0 \end{pmatrix}$$

$$K_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{0.1}{6} \begin{pmatrix} 1 + 1.05 \times 4 + 1.11 \\ 0 \end{pmatrix} = \begin{pmatrix} 1.105167 \\ 0 \end{pmatrix}$$

Implicit scheme

$$x_1 = x_0 + \frac{h}{2} \left(A x_0 + A x_1 \right)$$

$$\left(I - \frac{h}{2} A \right) x_1 = \left(I + \frac{h}{2} A \right) x_0$$

$$x_1 = \begin{pmatrix} 0.95 & -0.15 \\ 0 & 0.9 \end{pmatrix}^{-1} \begin{pmatrix} 1.05 & 0.15 \\ 0 & 1.1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{0.855} \begin{vmatrix} 0.9 & 0.15 \\ 0 & 0.95 \end{vmatrix} \begin{vmatrix} 1.05 & 0.15 \\ 0 & 1.1 \end{vmatrix}$$

$$= \begin{pmatrix} 1.105263 \\ 0 \end{pmatrix}$$