1) F07

a) $y = e^{\alpha x}$, $\alpha > 1$

$K = \left|\frac{x y'}{y}\right| = \left|\frac{x \alpha e^{\alpha x}}{e^{\alpha x}}\right| = |\alpha x|$, $f(x) = \alpha x$, $f'(x) = \alpha > 0$

$\Rightarrow f'(x)$ strikt monoton $\nearrow$ $\Rightarrow$ extremum am Rand des intervals $[0,1]$

$\lim_{x \downarrow 0} f(x) = 1$; $\lim_{x \uparrow 1} f(x) = \alpha$ $\Rightarrow \max_{x \in [0,1]} K(x) = \alpha$

Rel. Fehler: 0,1% $\Rightarrow \underbrace{0{,}001}_{0{,}1\%} \cdot \underbrace{\alpha}_{\max K} < \underbrace{0{,}01}_{1\%}$ $\Leftrightarrow \alpha < 10 \Rightarrow \alpha < (10)$

$\Rightarrow \underline{\underline{\alpha \in [1, 10]}}$

b)

```
% n=... gegeben
u=(1:n^2);
for i=1:n
   A(i,:)=u((i-1)*n+1 : i*n);
end
cond(A)
```

ii)

```
b=(1:n)';
[L,U,P]=lu(A);
b=P*b;
y=L\b;
x=U\y;
```

② F07

L^∞ norm? anstatt L^2

a) $A = D + L + R$, $D = \begin{pmatrix} \alpha & 0 \\ 0 & 3 \end{pmatrix}$, $L = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $R = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$

$T = -D^{-1}(L+R) = -\begin{pmatrix} 1/\alpha & 0 \\ 0 & 1/3 \end{pmatrix}\begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix} = -\begin{pmatrix} 0 & 2/\alpha \\ 1/3 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -2/\alpha \\ -1/3 & 0 \end{pmatrix}$

$\det(T - \lambda I) = \begin{vmatrix} -\lambda & -2/\alpha \\ -1/3 & -\lambda \end{vmatrix} = 0 \iff \lambda^2 - \frac{2}{3\alpha} = 0 \Rightarrow \lambda_{1,2} = \pm\sqrt{\frac{2}{3\alpha}}$

Kondition: $\max|\lambda_{1,2}| < 1 \iff \left|\sqrt{\frac{2}{3\alpha}}\right| < 1 \iff \frac{2}{3\alpha} < 1 \iff \frac{1}{\alpha} < \frac{3}{2} \Rightarrow \underline{\underline{\alpha > \frac{2}{3}}}$?

b) $Dx^{k+1} = -(L+R)x^k + b$, $x^0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$x^{k+1} = \underbrace{-D^{-1}(L+R)}_{T}x^k + D^{-1}b$, $T(\alpha=3) = \begin{pmatrix} 0 & -2/3 \\ -1/3 & 0 \end{pmatrix}$, $D^{-1}(\alpha=3) = \begin{pmatrix} 1/3 & 0 \\ 0 & 1/3 \end{pmatrix}$

$\Rightarrow x^1 = \begin{pmatrix} 0 & -2/3 \\ -1/3 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1/3 & 0 \\ 0 & 1/3 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -2/3 \\ -1/3 \end{pmatrix} + \begin{pmatrix} 1/3 \\ 0 \end{pmatrix} = \underline{\underline{\begin{pmatrix} -1/3 \\ -1/3 \end{pmatrix}}}$

c) $\underbrace{\|e\|_2^k}_{\text{Fehler nach k-Iteration}} \leq \|T\|_2^k \underbrace{\|e\|_2^0}_{\text{Anfangsfehler}}$

$\Rightarrow \underbrace{\|T\|_k^2}_{= e^{k \cdot \ln(\|T\|)}} \geq \underbrace{\frac{\|e\|^k}{\|e\|^0}}_{10^{-6}} \Rightarrow e^{k \cdot \ln(\|T\|)} \geq 10^{-6}$; $\|T\|_2 := \sqrt{\max(\text{eig}(T^T T))}$

$T^T T = \begin{pmatrix} 0 & -1/3 \\ -2/3 & 0 \end{pmatrix}\begin{pmatrix} 0 & -2/3 \\ -1/3 & 0 \end{pmatrix} = \begin{pmatrix} 1/9 & 0 \\ 0 & 4/9 \end{pmatrix}$, $\lambda_1 = 1/9$, $\lambda_2 = 4/9$

$\Rightarrow \max(\text{eig}(T^T T)) = \frac{4}{9} \approx 0{,}\bar{4} = \lambda_2$

$\Rightarrow \|T\|_2 = \sqrt{\lambda_2} = \frac{2}{3}$

$\Rightarrow k \cdot \ln\left(\frac{2}{3}\right) \geq \ln(10^{-6})$

$\iff k \geq \frac{\ln(10^{-6})}{\ln(2/3)} \approx 34{,}0732 \Rightarrow$ $\underline{\underline{k = 35 \text{ Iterationen}}}$

③ $x = a^x := f(x)$; $a > 1$

a) - Selbstabbildung von $f(x) = a^x$

$f(x)$ strikt monoton steigend $\Rightarrow$ extrema am Rand des intervals $[0,2]$:

$\begin{cases} f(0) = 1 & : \text{minimum} \\ f(2) = a^2 > 1 & : \text{maximum } (a>1) \end{cases} \Rightarrow \begin{cases} f(0) \geq 0 & ① \\ f(2) \leq 2 & ② \end{cases}$ weil $e^{2\cdot\ln(a)} \leq 2$

$\Leftrightarrow \ln(a) \leq \frac{\ln(2)}{2}$

$\Leftrightarrow a \leq e^{\ln(2)/2} = 1{,}41 = \sqrt{2}$

① + ② $\Rightarrow$ selbstabbildung ok

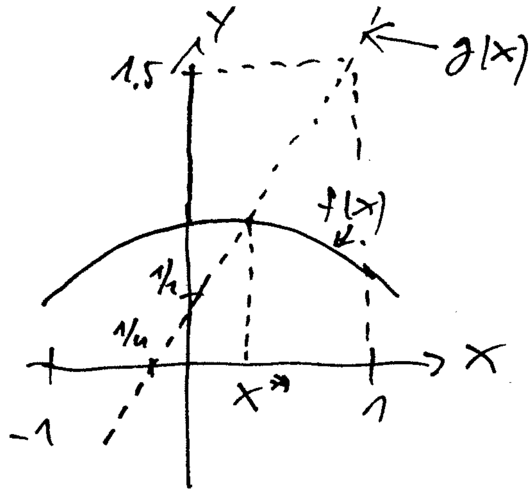
- $\exists C < 1$, $|f(x) - f(y)| < C|x-y|$, $\forall x, y \in I = [0,2]$

$f \in C^1(I) \Rightarrow C = \sup_{\substack{x\in I \\ a}} |f'(x)| = \sup_{\substack{x\in I \\ a}} \{|(e^{x\cdot\ln(a)})'|\} = \sup_{\substack{x\in I \\ a}} (e^{x\cdot\ln(a)} \cdot \ln(a))$

$= e^{2\cdot\ln(a)} \cdot \ln(a) = 0{,}691 < 1 \Rightarrow$ ok.

für $x = 2, a = \sqrt{2}$
$\Rightarrow$ maximal value

b) $f(x) = \cos(x)^2$, $g(x) = 2x + \frac{1}{2}$



gesucht: $f(x) = g(x) \Leftrightarrow f(x) - g(x) = 0$

$\Leftrightarrow \underbrace{\cos^2(x) - 2x - \frac{1}{2}}_{F(x)} = 0$

gesucht: $x^* \in [0,1]$

Newton-Verfahren: $x^{k+1} = x^k - \frac{F(x^k)}{F'(x^k)} = x^k - \frac{\cos^2(x^k) - 2x^k - 1/2}{-2\cdot\cos(x^k)\cdot\sin(x^k) - 2}$

wähle: $x^0 = 0{,}5$

$x^1 = 0{,}2431$

$x^2 = 0{,}2253$

$x^3 = 0{,}2250 \Rightarrow$ Lösung $x^* = 0{,}225$

$f(0{,}225) = 0{,}95022$; $g(0{,}225) = 0{,}950 \Rightarrow$ ok 3 Ziffern genau

$\Rightarrow x^* = (0{,}225 ; 0{,}950)$

④ F07

i) $b_i := \frac{1}{h_i} = 3 \quad (h = \frac{1}{3})$; $b_1 = \dots = b_6 = 3$ [n-1 Werte] ; $n = 7$ Punkte

$a_i := 2(b_i + b_{i+1}) = 2 \cdot 6 = 12$; $a_1 = \dots = a_5 = 12$ [n-2 Werte]

$c_i := \frac{f_{i+1} - f_i}{h_i^2} \Rightarrow$

$c_1 = (f_2 - f_1) \cdot 9 = 9$ $\qquad c_4 = (f_5 - f_4) \cdot 9 = -9$

$c_2 = (f_3 - f_2) \cdot 9 = 9$ $\qquad c_5 = (f_6 - f_5) \cdot 9 = -9$

$c_3 = (f_4 - f_3) \cdot 9 = -18$ $\qquad c_6 = (f_7 - f_6) \cdot 9 = 18$

$d_1 := 2 \cdot c_1 + \frac{h_1}{h_1 + h_2}(c_1 + c_2) = 18 + \frac{1}{2}(c_1 + c_2) = 18 + 9 = 27$

$d_2 := 3(c_1 + c_2) = 54$; $d_3 := 3(c_2 + c_3) - 27$; $d_4 := 3(c_3 + c_4) = -81$;

$d_5 := 3(c_4 + c_5) = -54$; $d_6 := 3(c_5 + c_6) = 27$; $d_7 = 2 \cdot c_6 + \frac{1}{2}(c_6 + c_5) = 40{,}5$

$$\Rightarrow \begin{bmatrix} 12 & 3 & & & & 3 \\ 3 & 12 & 3 & & & \\ & 3 & 12 & 3 & & \\ & & 3 & 12 & 3 & \\ & & & 3 & 12 & 3 \\ 8 & 2 & & & & 2 \end{bmatrix} \begin{bmatrix} f_1' \\ f_2' \\ f_3' \\ f_4' \\ f_5' \\ f_6' \end{bmatrix} = \begin{bmatrix} 27 \\ 54 \\ -27 \\ -81 \\ -54 \\ 54 \end{bmatrix}$$

$8f_1' + 2f_2' + 2f_{n-1}' = \frac{6}{h}(f_2 - f_{n-1})$

ii) A = [3 6 0 0 0 0; 3 12 3 0 0 0; 0 3 12 3 0 0; 0 0 3 12 3 0;
0 0 0 3 12 3; 0 0 0 0 6 3]';

b = [27; 54; -27; -81; -54; 27]; % oder b = [27 54 -27 -81 -54 27]';

f = A\b;

(5) F07 $\quad I = \int_0^\lambda e^x \, dx \, , \lambda > 0$

a) $S_0 = \frac{1}{2}[f(0) + f(\lambda)] = \frac{1}{2}[1 + e^\lambda]$; $T_0 = \overbrace{\lambda}^{\text{Schritt } h} \cdot S_0 = \frac{\lambda}{2}[1 + e^\lambda]$

$S_1 = S_0 + f(0 + \underbrace{\tfrac{\lambda}{2}}_{h/2}) = \frac{1}{2}[1 + e^\lambda] + e^{\lambda/2}$; $T_1 = \underbrace{\frac{\lambda}{2}}_{h/2} \cdot S_1 = \frac{\lambda}{2}\left[\frac{1+e^\lambda}{2} + e^{\lambda/2}\right]$

$S_2 = S_1 + f(0 + \frac{\lambda}{4}) + f(0 + \frac{3\lambda}{4})$

$= \frac{1}{2}[1 + e^\lambda] + e^{\lambda/2} + e^{\lambda/4} + e^{3\lambda/4}$; $T_2 = \frac{\lambda}{4} \cdot S_2 = \frac{\lambda}{4} \cdot \left[\frac{1+e^\lambda}{2} + e^{\lambda/2} + e^{\lambda/4} + e^{3\lambda/4}\right]$

b) $T_2 = \frac{1}{4}\left[\frac{1+e}{2} + e^{1/2} + e^{1/4} + e^{3/4}\right] = 1{,}7272\ 2190$

Exakte Lösung: $\int_0^1 e^x \, dx = e^1 - e^0 = e - 1 = 1{,}7182818$

Fehler: $\Delta f = 0{,}00894$

c) $R_{00} = T_0 = 1{,}8591409$

$R_{11} = \frac{R_{10} - 4^{-1} R_{00}}{1 - 4^{-1}} = 1{,}7188610$

$R_{10} = T_1 = 1{,}7539310$

$R_{21} = \frac{R_{20} - 4^{-1} R_{10}}{1 - 4^{-1}} = 1{,}7183188$

$R_{20} = T_2 = 1{,}7272219$

$R_{21} = \frac{R_{21} - 4^{-2} \cdot R_{11}}{1 - 4^{-2}} =$

$R_{21} = 1{,}7177766$

Fehler: $\Delta f = 0{,}00051$

⑥ F07

a) i) 2-Stufe, ~~im~~plizit, Heun-Verfahren

ii) $K_1 = f(y)$

$K_2 = f(h, y_k + h \cdot f(y_k))$

$$\Rightarrow \underline{\underline{y^{k+1} = y^k + \frac{h}{2}\left[f(y^k) + f(y^k + h \cdot f(y^k))\right]}}$$

b) i) $y_1 = x$, $y_2 = \dot{x}$ $\Rightarrow$ $\left.\begin{array}{l} \dot{y}_2 = \ddot{x} = -\dot{x} - x = -y_2 - y_1 \\ \dot{y}_1 = y_2 \end{array}\right\} \Rightarrow \frac{d}{dt}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$

oder $\frac{d}{dt} y = \underbrace{\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}}_{A} y$

Ersetzen im a)

$$y^{k+1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} y^k + \frac{h}{2}\left[\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} y^k + \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \cdot \left\{\cancel{\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}} y^k + h \cdot \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} y^k\right\}\right]$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} y^k + \begin{pmatrix} 0 & h/2 \\ -h/2 & -h/2 \end{pmatrix} y^k + \begin{pmatrix} 0 & h/2 \\ -h/2 & -h/2 \end{pmatrix} \left\{\cancel{\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}} y^k + \begin{pmatrix} 0 & h \\ -h & -h \end{pmatrix} y^k\right\}$$

$$= \begin{pmatrix} 1 & h/2 \\ -h/2 & 1-\frac{h}{2} \end{pmatrix} y^k + \begin{pmatrix} 0 & -h/2 \\ -h/2 & -\frac{h}{2} \end{pmatrix} y^k + \begin{pmatrix} -h^2/2 & -h^2/2 \\ h^2/2 & 0 \end{pmatrix} y^k$$

$$= \begin{pmatrix} 1 - \frac{h^2}{2} & \frac{h}{2} + \frac{h}{2} - \frac{h^2}{2} \\ -\frac{h}{2} - \frac{h}{2} + \frac{h^2}{2} & 1 - \frac{h}{2} - \frac{h}{2} \end{pmatrix} y^k = \underline{\underline{\underbrace{\begin{pmatrix} 1 - \frac{h^2}{2} & h - \frac{h^2}{2} \\ \frac{h^2}{2} - h & 1 - h \end{pmatrix}}_{A(h)}}} \cdot y^k$$

ii) $h = 1 \Rightarrow A(h) = \begin{pmatrix} 1/2 & +1/2 \\ -1/2 & 0 \end{pmatrix}$, $A^T A = \begin{pmatrix} 1/2 & 1/4 \\ 1/4 & 1/4 \end{pmatrix}$

Eigenvalue von $A^T A$: $\begin{vmatrix} 1/4 - \lambda & 1/4 \\ 1/4 & 1/2 - \lambda \end{vmatrix} = 0 \Leftrightarrow (\frac{1}{4} - \lambda)(\frac{1}{2} - \lambda) - \frac{1}{16} = 0$

$\Leftrightarrow \lambda^2 - \frac{3}{4}\lambda + \frac{1}{16} = 0$, $\Delta = 5/16$

$\Rightarrow \lambda_{1,2} = \underline{3/4 \pm \sqrt{5/16}}$ $\begin{array}{l} \nearrow \lambda_1 = 0{,}654508 \\ \searrow \lambda_2 = 0{,}095492 \end{array}$

$\Rightarrow \|A(h)\|_2 := \sqrt{\max(\mathrm{eig}(A^T A))}$

$= \sqrt{\lambda_1} = \underline{\underline{0{,}80902}}$; $\|y^{k+1}\| = \overbrace{\|A(h)\|}^{<1}\|y^k\|$

$\Rightarrow \lim \ldots$