ETHZ, BSc D-MAVT
Lösungen Prüfung Frühjahr 08
Numerische Mathematik
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1. **a)** $f_0 = f(x_0) = 0.3333,\ f_1 = f(x_1) = 0.3938,\ f_2 = f(x_2) = 0.5496$

$$\begin{aligned} l_0(x) &= \frac{(x-\frac{1}{3})(x-1)}{\frac{-1}{3}\cdot(-1)} = 3(x-\frac{1}{3})(x-1) \\ l_1(x) &= \frac{(x)(x-1)}{\frac{1}{3}\cdot\frac{-2}{3}} = -\frac{9}{2}(x)(x-1) \\ l_2(x) &= \frac{(x)(x-\frac{1}{3})}{1\cdot\frac{2}{3}} = \frac{3}{2}(x)(x-\frac{1}{3}) \end{aligned}$$

$$p(x) = 3(x-\frac{1}{3})(x-1)f_0 - \frac{9}{2}(x)(x-1)f_1 + \frac{3}{2}(x)(x-\frac{1}{3})f_2$$

b)

$$l_0(\frac{2}{3}) = -1/3,\ l_1(\frac{2}{3}) = 1,\ l_2(\frac{2}{3}) = 1/3,$$

$$p(\frac{2}{3}) = -f_0/3 + f_1 + f_2/3 \approx 0.4659$$

c) Es gilt

$$f(x) - p_2(x) = \frac{f^{(3)}(\xi)}{(n+1)!}(x)(x-\frac{1}{3})(x-1)$$

$$|f'''(\xi)| = \frac{1}{24}e^{\frac{\xi}{2}} \leq \frac{1}{24}e^{\frac{1}{2}} = 0.0686967$$

$$|(x)(x-\frac{1}{3})(x-1)| \leq 0.08$$

$$\Rightarrow |p(x) - f(x)| \leq \frac{0.08}{6}0.0686967 = 9.15956e-4$$

(Der maximale Fehler beträgt 7.066707e-4.)

2.

$$J = \begin{pmatrix} 3x_1^2 & 1 \\ x_2 & x_1 - 6x_2 \end{pmatrix} \quad f(2,1) = (-5,\, 0)' \quad J(2,1) = \begin{pmatrix} 12 & 1 \\ 1 & -4 \end{pmatrix}$$

$$J^{-1}(2,1) = \frac{-1}{49} \begin{pmatrix} -4 & -1 \\ -1 & 12 \end{pmatrix}$$

$$x_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} - \frac{-1}{49} \begin{pmatrix} -4 & -1 \\ -1 & 12 \end{pmatrix} \begin{pmatrix} -5 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{118}{49} \\ \frac{54}{49} \end{pmatrix}$$

a)
```
function[x]=quasinewtion(f,Df,x0,tol, maxit)
  [L, U]=lu(Df(x0));
  x_alt=x0;
  for i=1:maxit
   %Quasinewtonschritt unter Verwendung von L und U
   x_neu=x_alt-L\U\f(x_alt)
   norm(x_alt-x_neu)
   if(norm(x_alt-x_neu)<tol*(norm(x_neu)+1))
    x=x_neu;
    break
   end
   x_alt=x_neu
  end
```

3. a) $\omega = exp(-\pi i/4) = -i$

$$W = \frac{1}{4} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{pmatrix}$$

Also

$$\begin{aligned}
\tilde{c}_0 &= \frac{1}{4}(f_0 + f_1 + f_2 + f_3) = \cancel{\frac{5}{4}} \quad 0.725 \\
\tilde{c}_1 &= \frac{1}{4}(f_0 - if_1 - f_2 + if_3) = \frac{1}{2} \cancel{-} \frac{1}{4}i \quad -\frac{1}{4} \quad -0.175 \\
\tilde{c}_2 &= \frac{1}{4}(f_0 - f_1 + f_2 - f_3) = \cancel{-\frac{5}{4}} \quad 0.275 \\
\tilde{c}_3 &= \frac{1}{4}(f_0 + if_1 - f_2 - if_3) = \frac{1}{2} + \frac{1}{4}i \quad -\frac{1}{4} \quad +0
\end{aligned}$$

b) Somit

$a_0 = \tilde{c}_0 = 0.725$

$$\begin{aligned}
a_0 &= \frac{1}{4}(f_0 + \cancel{f_1 +} f_2 + f_3) = \frac{5}{4} \\
a_1 &= (\tilde{c}_1 + \tilde{c}_3) = \frac{1}{2}(f_0 - f_2) = 1 \quad - \\
b_1 &= i(\tilde{c}_1 - \tilde{c}_3) = \frac{1}{2}(f_1 - f_3) = \frac{1}{2} \\
a_2 &= \tilde{c}_2 = \frac{1}{4}(f_0 - f_1 + f_2 - f_3) = \cancel{-\frac{5}{4}} \quad 0.275
\end{aligned}$$

(Für b) darf auch ein Gleichungssystem gelöst werden.)

4. a)

$$
\begin{aligned}
f(x) = 1 \quad &: \quad \int_0^1 x^{\frac{1}{2}}\,dx = \frac{2}{3} \;\Rightarrow w_1 + w_2 = \frac{2}{3} \\
f(x) = x \quad &: \quad \int_0^1 x^{\frac{3}{2}}\,dx = \frac{2}{5} \;\Rightarrow w_1 x_1 + w_2 = \frac{2}{5} \\
f(x) = x^2 \quad &: \quad \int_0^1 x^{\frac{5}{2}}\,dx = \frac{2}{7} \;\Rightarrow w_1 x_1^2 + w_2 = \frac{2}{7}
\end{aligned}
$$

$$
\begin{aligned}
w_1(x_1 - 1) &= \frac{-4}{15},\; w_1 x_1 (x_1 - 1) = \frac{-4}{35} \\
x_1 &= \frac{-4}{35}\frac{15}{-4} = \frac{3}{7},\; w_1 = \frac{-4}{15}\frac{7}{-4} = \frac{7}{15} \\
w_2 &= \frac{2}{3} - \frac{7}{15} = \frac{3}{15} = \frac{1}{5}
\end{aligned}
$$

b) $\hat{I} = \frac{7}{15}\frac{\sqrt{3/7}}{10/7} + \frac{1}{5}\frac{1}{2} \approx 0.31385353243127$

$$\hat{I} = \frac{7}{15}\underbrace{\frac{1}{1+\frac{3}{7}}}_{\frac{7}{10}} + \frac{1}{5}\frac{1}{2} = \frac{49}{150} + \frac{1}{10} = \frac{64}{150} = \frac{32}{75}$$

5. a) $x_1 = x, \quad x_2 = \dot{x}.$

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ -x_1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

- $\vartheta = 0$:

$$z^1 = z^0 + Az_0 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$k_1 = A(z_0 + \frac{h}{2}k_1) =$

$z^1 = z^0 + h k_1$

- $\vartheta = 0.5$:

$$z^1 = z^0 + \frac{1}{2}A(z_0 + z_1)$$

$$(1 - \frac{1}{2}A)z^1 = (1 + \frac{1}{2}A)z_0$$

$$\begin{pmatrix} 1 & -1/2 \\ 1/2 & 1 \end{pmatrix} z_1 = \begin{pmatrix} 1 & +1/2 \\ -1/2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1/2 \end{pmatrix}$$

$$z_1 = \frac{1}{5} \begin{pmatrix} 4 & +2 \\ -2 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ -1/2 \end{pmatrix} = \begin{pmatrix} 3/5 \\ -4/5 \end{pmatrix}$$

$z^1 = \begin{pmatrix} \frac{3}{5} \\ -\frac{4}{5} \end{pmatrix}$

- $\vartheta = 1$:

$$z^1 = z^0 + Az_1$$

$$(1 - A)z^1 = z_0$$

$$\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} z_1 = z_0$$

$$z_1 = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1/2 \\ -1/2 \end{pmatrix}$$

b) • $\vartheta = 0$:

$$\begin{aligned} z^1 &= z^0 + Az_0 \\ z^1 &= (I + A)z_0 \\ B_0 &= \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \\ B_0^T B_0 &= \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \\ (z^n)^T z^n &= 2(z^{n-1})^T z^{n-1} \\ ||z_n||_2 &\to \infty \ (n \to \infty) \end{aligned}$$

- $\vartheta = 0.5$:

$$\begin{aligned} z^1 &= z^0 + \cancel{\frac{1}{2}A(z_0 + z_1)} \quad (I - \frac{1}{2}A)^{-1} A z_0 \\ z^1 &= (I - \frac{1}{2}A)^{-1}(I + \frac{1}{2}A)z_0 \\ B_{1/2} &= \frac{4}{5} \begin{pmatrix} 3/4 & 1 \\ -1 & 3/4 \end{pmatrix} = \begin{pmatrix} 3/5 & 4/5 \\ -4/5 & 3/5 \end{pmatrix} \\ B_{1/2}^T B_{1/2} &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ (z^n)^T z^n &= (z^{n-1})^T z^{n-1} \\ ||z_n||_2 &= ||z_0||_2 (n \to \infty) \end{aligned}$$

- $\vartheta = 1$:

$$
\begin{aligned}
z^1 &= z^0 + Az_1 \\
z^1 &= (I - A)^{-1} z_0 \\
B_1 &= \frac{1}{2}\begin{pmatrix} 1 & +1 \\ -1 & 1 \end{pmatrix} \\
B_1^T B_1 &= \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix} \\
(z^n)^T z^n &= 1/2 (z^{n-1})^T z^{n-1} \\
||z_n||_2 &\rightarrow 0 \ (n \rightarrow \infty)
\end{aligned}
$$

6. **a)** Gleichung fuer $u_1 = u(x_1, y_1)$

$$4u_1 - u_2 - u_3 - \frac{1}{3} - 0 = \frac{1}{9} \cdot \frac{1}{3}$$

Gleichung fuer $u_2 = u(x_2, y_2)$

$$4u_2 - u_1 - 1 - 2/3 - 2/3 = \frac{1}{9} \cdot \frac{1}{3}$$

Gleichung fuer $u_3 = u(x_3, y_3)$

$$4u_3 - u_1 - 2/3 - 0 - 1/3 = \frac{1}{9} \cdot \frac{2}{3}$$

Also

$$\begin{pmatrix} 4 & -1 & -1 \\ -1 & 4 & 0 \\ -1 & 0 & 4 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \frac{2}{27} \begin{pmatrix} 5 \\ 32 \\ 14 \end{pmatrix}$$

b) Die Matrix ist symmetrisch und strikt diagonal dominant mit positiven Diagonalelementen.