„Numerische Mathematik"
He 04

① $I = \int_0^{\pi/4} \cos 2x \, dx$

Exact

$I = \left[\frac{1}{2}\sin 2x\right]_0^{\pi/4} = \frac{1}{2}\left(\sin\frac{\pi}{2} - \sin 0\right) = \frac{1}{2}$ ②

a i) $I = \int_0^{\pi/4} \cos 2x \, dx = \int_{-1}^{+1} \frac{\pi}{8} \cos 2\left(\frac{\pi}{8}(1+\xi)\right) d\xi$

$\approx \sum_{m=1}^{2} \frac{\pi}{8} \cos 2\left(\frac{\pi}{8}(1+\xi_m)\right) w_m$

$\xi_1 = \frac{1}{\sqrt{3}}$, $\xi_2 = -\frac{1}{\sqrt{3}}$ $w_1 = w_2 = 1$

$I \approx 0.499236$ $\Delta e / I = 1.528 \times 10^{-3}$ ②

a ii) $I = \int_0^{\pi/8} \cos 2x \, dx + \int_{\pi/8}^{\pi/4} \cos 2x \, dx$

$= \int_{-1}^{+1} \frac{\pi}{16} \cos \frac{2\pi}{16}(1+\xi) \, d\xi + \int_{-1}^{+1} \frac{\pi}{16} \cos 2\left(\frac{3\pi}{16} + \frac{\pi}{16}\xi\right) d\xi$

$\approx \frac{\pi}{16}\left(\underbrace{\sum_{m=1}^{2} w_m \cos\frac{2\pi}{16}(1+\xi_m)}_{0.3835} + \underbrace{\sum_{m=1}^{2} w_m \cos\frac{2\pi}{16}(3+\xi_m)}_{0.746}\right)$

$= 0.49995$ $\Delta e / I = 8.98 \times 10^{-5}$ ②

b) $p(x) = a + bx + cx^2 + dx^3$

$I = \int_{-1}^{+1} p(x) \, dx = \left[ax + \frac{bx^2}{2} + \frac{cx^3}{3} + \frac{dx^4}{4}\right]_{-1}^{+1} = \frac{2c}{3} + 2a$

$I \approx \sum_{m=1}^{2} p(x_m) w_m = a + \frac{b}{\sqrt{3}} + \frac{c}{3} + \frac{d}{(\sqrt{3})^3} + a - \frac{b}{\sqrt{3}} + \frac{c}{3} - \frac{d}{(\sqrt{3})^3} = \frac{2c}{3} + 2a$

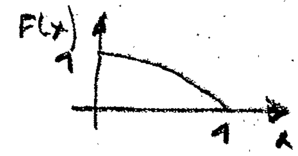
②

② ∴ Option 1 $x = 1 - x^2$ option 2 $x = \frac{1 + x - x^2}{2}$

<u>Option 1</u>

$F(x) = 1 - x^2$

$I = [0,1]$



$F(0) = 1$, $F(1) = 0$

$F(0), F(1) \in I$ ∴ ok selbstandig

$\exists\, C < 1$ s.d. $|F(x') - F(x'')| \le C|x' - x''| \;\forall\; x', x'' \in I$ (Kontraktion)

$F \in C^1(I) \Rightarrow C = \sup_{x \in I} |F'(x)| = \sup_{x \in I} |-2x| = 2 > 1$ ∴ Fails

②

<u>Option 2</u>

$F(x) = \frac{1 + x - x^2}{2}$

$I = [0,1]$

$F(0) = 1/2 \in I$, $F(1) = 1/2 \in I$, $F(1/2) = 5/8 \in I$ ∴ ok selbstandig

$\exists\, C < 1$ s.d. $|F(x') - F(x'')| \le C|x' - x''| \;\forall\; x', x'' \in I$ (Kontraktion)

$F \in C^1(I) \Rightarrow C = \sup_{x \in I} |F'(x)| = \sup_{x \in I} \left|\frac{1}{2}(1 - 2x)\right| = 1/2 < 1$ ∴ ok

②

Use option 2

b) $x_{n+1} = F(x_n)$ $x_0 = 1/2$, $x_1 = \frac{1 + 1/2 - (1/2)^2}{2} = \frac{10}{16}$, $x_2 = \frac{1 + 10/16 - (10/16)^2}{2}$

≈ 0.61718 ok ②

c) $\frac{|x - x_k|}{|x_{j+1} - x_j|} \le \frac{C^{k-j}}{1 - C}$ sei $j = 0$, $|x_1 - x_0| \le |I| = 1$

$|x - x_k| < 10^{-6}$

$10^{-6} \le \frac{(1/2)^k}{1/2}$ $k \geq \frac{\ln(1/2 \cdot 10^{-6})}{\ln(1/2)} = 21$ iterations ②

also allow for their chosen x_1, x_0 values.

③ a)

$$\begin{pmatrix} 0 & 1 \\ 2.5 & 1 \\ 5 & 1 \\ 7.5 & 1 \\ 10.0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} - \begin{pmatrix} 0.03 \\ 0.17 \\ 0.37 \\ 0.44 \\ 0.56 \end{pmatrix} = 0$$

$$\underline{\underline{A}} \; \underline{x} - \underline{d} = \underline{0}$$

Gauss Method:

- $\underline{\underline{A}}^T \underline{\underline{A}} \, \underline{x} = \underline{\underline{A}}^T \underline{d}$ ①

$$\underline{\underline{A}}^T \underline{\underline{A}} = \begin{pmatrix} 0 & 2.5 & 5 & 7.5 & 10.0 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 2.5 & 1 \\ 5 & 1 \\ 7.5 & 1 \\ 10.0 & 1 \end{pmatrix} = \begin{pmatrix} 187.5 & 25.0 \\ 25.0 & 5.0 \end{pmatrix}$$ ①

$$\underline{\underline{A}}^T \underline{d} = \begin{pmatrix} 0 & 2.5 & 5 & 7.5 & 10.0 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0.03 \\ 0.17 \\ 0.37 \\ 0.44 \\ 0.56 \end{pmatrix} = \begin{pmatrix} 11.7750 \\ 1.570 \end{pmatrix}$$ ①

- $\underline{\underline{A}}^T \underline{\underline{A}} \, \underline{x} = \underline{\underline{A}}^T \underline{d}$

$$\underline{x} = \begin{pmatrix} 0.0532 \\ 0.0480 \end{pmatrix}$$ ① $C = 0.0532\Theta + 0.048$

b) $\underline{\underline{Q}} = \frac{1}{9} \begin{pmatrix} 1 & 0 & 4 & 8 & 0 \\ 0 & 9 & 0 & 0 & 0 \\ 4 & 0 & 7 & -4 & 0 \\ 8 & 0 & -4 & 1 & 0 \\ 0 & 0 & 0 & 0 & 9 \end{pmatrix}$ $\quad \underline{\underline{R}} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$ $\quad \underline{c} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{pmatrix}$

$$\underline{d} = \underline{\underline{Q}}^T \underline{c} = \frac{1}{9} \begin{pmatrix} 1 & 0 & 4 & 8 & 0 \\ 0 & 9 & 0 & 0 & 0 \\ 4 & 0 & 7 & -4 & 0 \\ 8 & 0 & -4 & 1 & 0 \\ 0 & 0 & 0 & 0 & 9 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{pmatrix} = \frac{1}{9} \begin{pmatrix} 45 \\ 18 \\ 9 \\ 0 \\ 45 \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \\ 1 \\ 0 \\ 5 \end{pmatrix} \Big\} d_0$$ ②

$$\therefore \underline{x} = R_0^{-1} d_0 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 5 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad ②$$

Check using normal method

↓ Not required by students

$$A = QR = \frac{1}{9} \begin{pmatrix} 1 & 0 & 4 & 8 & 0 \\ 0 & 9 & 0 & 0 & 0 \\ 4 & 0 & 7 & -4 & 0 \\ 8 & 0 & -4 & 1 & 0 \\ 0 & 0 & 0 & 0 & 9 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} = \frac{1}{9} \begin{pmatrix} 1 & 2 \\ 0 & 9 \\ 4 & 8 \\ 8 & 16 \\ 0 & 0 \end{pmatrix}$$

$$\underline{\underline{A}}^T \underline{\underline{A}} = \frac{1}{81} \begin{pmatrix} 1 & 0 & 4 & 8 & 0 \\ 2 & 9 & 8 & 16 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 9 \\ 4 & 8 \\ 8 & 16 \\ 0 & 0 \end{pmatrix} = \frac{1}{81} \begin{pmatrix} 81 & 162 \\ 162 & 405 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}$$

$$A^T c = \frac{1}{9} \begin{pmatrix} 1 & 0 & 4 & 8 & 0 \\ 2 & 9 & 8 & 16 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{pmatrix} = \frac{1}{9} \begin{pmatrix} 45 \\ 108 \end{pmatrix} = \begin{pmatrix} 5 \\ 12 \end{pmatrix}$$

$$\underline{x} = (A^T A)^{-1} (A^T c) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Periodic Spline

x_i	0	1/3	2/3	1
f_i	-1	0	1	-1

n-2 equations are

$b_1 f_1' + a_1 f_2' + b_2 f_3' = d_2$ ①

$b_2 f_2' + a_2 f_3' + b_3 f_4' = d_3$

end conditions are

$8 f_1' + 2 f_2' + 2 f_3' = \frac{6}{h}(f_2 - f_3)$ and $f_4' = f_1'$ ②

$$\Rightarrow \begin{pmatrix} 8 & 2 & 2 \\ b_1 & a_1 & b_2 \\ b_3 & b_2 & a_2 \end{pmatrix} \begin{pmatrix} f_1' \\ f_2' \\ f_3' \end{pmatrix} = \begin{pmatrix} 6/h\,(f_2 - f_3) \\ d_2 \\ d_3 \end{pmatrix}$$ ①

$b_1 = b_2 = b_3 = \frac{1}{h} = 3$

$a_1 = 2(b_1 + b_2) = 12$, $a_2 = 2(b_2 + b_3) = 12$

$d_2 = 3(c_1 + c_2)$, $d_3 = 3(c_2 + c_3)$

$c_1 = \frac{f_2 - f_1}{h^2} = 9$

$d_2 = 54$

$c_2 = \frac{f_3 - f_2}{h^2} = 9$

$d_3 = -27$

$c_3 = \frac{f_4 - f_3}{h^2} = -18$

$$\begin{pmatrix} 8 & 2 & 2 \\ 3 & 12 & 3 \\ 3 & 3 & 12 \end{pmatrix} \begin{pmatrix} f_1' \\ f_2' \\ f_3' \end{pmatrix} = \begin{pmatrix} 18 \\ 54 \\ -27 \end{pmatrix} \leadsto \begin{pmatrix} f_1' \\ f_2' \\ f_3' \end{pmatrix} = \begin{pmatrix} -3 \\ 6 \\ -3 \end{pmatrix}$$ ① ①

Approximate at $x = 0.8$. Last interval $t = (0.8 - 2/3)/\frac{1}{3} = 0.4$

$a_0 = 1$, $a_1 = -1$ $\Rightarrow b_0 = -2$; $b_{-1} = -1$, $b_1 = $ [illegible] $\Rightarrow c_0 = -1$, $c_1 = 1$ $\Rightarrow d_0 = 2$

$Q(t = 0.4) = $ [illegible] ② 0.24

(5) $m\ddot{x} + c\dot{x} + kx = 0$

$x(0) = 1$

a) $\dot{x}(0) = x$ required intial condition. (1)

b) $\dot{x}(0) = 1$

$m = c = 1$, $k = 2$

$$\left.\begin{array}{l} \ddot{x} + \dot{x} + 2x = 0 \\ x(0) = 1 \\ \dot{x}(0) = 1 \end{array}\right\}$$ (1)

$u_1 = x$, $u_2 = \dot{x}$

$\dot{u}_1 = \dot{x} = u_2$, $\dot{u}_2 = \ddot{x} = -\dot{x} - 2x = -u_2 - 2u_1$

$$\begin{pmatrix} \dot{u}_1 \\ \dot{u}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -2 & -1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$ system of two first order equations (2)

$\dot{\underline{u}} = \underline{\underline{A}}\,\underline{u} = f(t,u)$

Apply Heun scheme

$K_1 = f(t_j, u_j)$

$K_2 = f(t_j + h, u_j + hK_1)$

$u_{j+1} = u_j + \frac{h}{2}(K_1 + K_2)$

0	0	0
1	1	0
	1/2	1/2

(2)

$$\underline{K_1} = \begin{pmatrix} 0 & 1 \\ -2 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$$

$$\underline{K_2} = \begin{pmatrix} 0 & 1 \\ -2 & -1 \end{pmatrix} \begin{pmatrix} 1+0.1 \\ 1-0.3 \end{pmatrix} = \begin{pmatrix} 0.7 \\ -2.9 \end{pmatrix}$$

$$x_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{0.1}{2} \begin{pmatrix} 1+0.7 \\ -3-2.9 \end{pmatrix} = \begin{pmatrix} 1.085 \\ 0.705 \end{pmatrix}$$ (2)

6) $\begin{array}{c|c} \theta & \theta \\ \hline & 1 \end{array}$

a) $\theta = 1$

$x_{n+1} = x_n + hK_1$

$K_1 = f(t_n+h, x_n + hK_1) = f(t_n+h, x_n + x_{n+1} - x_n) = f(t_n+h, x_{n+1})$

$x_{n+1} = x_n + hf(t_n+h, x_{n+1})$

$= x_n + hf(t_n+h, x_n + hf(t_n+h, x_{n+1}))$ (1)

Use Taylor series expansion for $f(t_n+h, x_{n+1})$

Set $n=0$, $t_0 = 0$

$f(h, x_1) = f(0, x_0) + h(\dot f(0, x_0) + \underbrace{f(h, x_1)}_{\text{use Taylor series again}} f'(0, x_0)) + O(h^2)$ (1)

$f(h, x_1) = f(0, x_0) + h(\dot f(0, x_0) + (f(0, x_0) + O(h))f'(0, x_0)) + O(h^2)$

$= f(0, x_0) + h(\dot f(0, x_0) + f(0, x_0)f'(0, x_0)) + O(h^2)$ (1)

$x_1 = x_0 + h(f(0, x_0)) + h^2(\dot f(0, x_0) + f(0, x_0)f'(0, x_0)) + O(h^3)$

Exact Taylor series expansion is:

$x(t=h) = x(0) + h\dot x(0) + \frac{h^2}{2!} + O(h^3)$

$\dot x = f(t, x) \quad \ddot x = \frac{\partial}{\partial t}(\dot x) = \partial_t f + \partial_x f \partial_t x = \partial_t f + \dot x \partial_x f = \partial_t f + f \partial_x f$ (1)

$x(t=h) = x(0) + hf(0, x(0)) + \frac{h^2}{2}(\dot f(0, x(0)) + f(0, x(0))f'(0, x(0))) + O(h^3)$

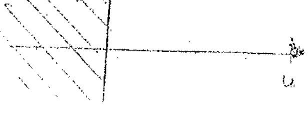
$|x(t=h) - x_n| = O(h^2) \Rightarrow$ Ordnung 1 (1)

b) $\theta = 1/2 \quad K_1 = f(t_n + h/2, x_n + \frac{K_1 h}{2}) \quad x_{n+1} = x_n + hK_1 = x_n + hf(t_n + \frac{h}{2}, \frac{x_n + x_{n+1}}{2})$ (1)

Model equation $\dot x = f(t, x) = \lambda x$, $\lambda \in \mathbb{C}$ (1)

$x_{n+1} = x_n + h\lambda\left(\frac{x_n + x_{n+1}}{2}\right) \Rightarrow x_{n+1} = \left(\frac{2 + h\lambda}{2 - h\lambda}\right)x_n$, $R(\mu) = \left(\frac{2 + \mu}{2 - \mu}\right)$

$|R(\mu)| = R(\mu)\overline{R(\mu)} \leq 1 \quad \mu = u + iv$ (1)



$(2+u)^2 + v^2 / ((2-u)^2 + v^2) \leq 1 \quad u \leq 0 \quad A: \{\mu \in \mathbb{C} : \mathrm{Re}(\mu) \leq 0\}$