Notes on Lectures on Algebraic Geometry

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1 Preamble

These are notes to accompany lectures for an introductory course in algebraic geometry intended for students who have recently completed a semester-long course in commutative algebra following Atiyah–Macdonald. We have thus assumed some familiarity with the basic results and notation of that book. Others have thought longer and more carefully than I have about how to teach algebraic geometry. Some of their books and course notes have been linked on the homepage, while many more can and should be found online. Some of the topics covered in these notes, particularly those belonging more properly to commutative algebra, were not presented in lecture. We have included them here for the sake of completeness.

2 What's been covered in the lectures

1. Affine varieties. Motivational questions and examples.
2. Nullstellensatz and its geometric interpretations.
3. Polynomial maps between affine varieties. Examples. "Dominant maps are those with injective pullback" and "closed embeddings are those with surjective pullback".
4. Geometric interpretation of localization as restricting away from the locus of a polynomial and adding a dummy variable representing its inverse.
5. Partitions of unity, regular functions on open subsets of affine varieties, definitions of k-sheaf, k-space
6. Varieties as separated prevarieties. Quasi-affine varieties are varieties. Projective varieties.
7. The standard affine open cover of projective space. The affine open cover of a projective variety as intersecting the affine cone with hyperplanes. Projective varieties are varieties (i.e., admit a finite affine open cover and are separated).
8. Affine cones and the homogeneous nullstellensatz. Projective closure of an affine variety. Examples.
9. Detailed discussion of the twisted cubic curve and its projective closure. Preliminary discussion of Groebner bases.
10. Discussion of Groebner bases, Buchberger criterion/algorithm, how to compute projective closures.

11. Rational maps, rational functions, basic properties.
12. Examples of rational maps.
13. Blow-ups.
14. Tangent cone, smoothness
15. More on smoothness, resolving curve singularities via blow-up
16. Algebraic groups (examples, affine implies linear)
17. Toric varieties
18. Convex geometry and toric varieties
19. Images of morphisms (Chevalley; projective implies proper)
20. Proofs from last time; dimension; uniformizers on curves
21. Extending morphisms from a curve minus a smooth point; applications
22. Curves up to birational equivalence; degrees of morphisms, divisors
23. More on Div, Pic, etc.
24. Group law on an elliptic curve
25. Differentials
26. Introduction to schemes
27. Sketch of proof of Hasse bound for elliptic curves

3 Introduction

Let k be a field. We shall mostly consider the case that k is algebraically closed, denoted $k = \overline{k}$.

3.1 Affine varieties

Definition 1. By an *affine k-variety*[1] X, or simply an *affine variety* when the field k is clear from context, we shall mean the set of solutions

$$\alpha := (\alpha_1, \dots, \alpha_n) \in k^n =: \mathbb{A}^n$$

in some finite number n of variables to a system of equations

$$f(\alpha) = 0 \quad (f \in S)$$

[1] Caution: we adopt the convention that a variety need not be irreducible.

defined by a collection

$$S \subset k[x_1, \dots, x_n]$$

of *polynomials* in the variables $x_1, \dots, x_n$ with coefficients in k, and with $f(\alpha)$ denoting the element of k obtained from the polynomial f by substituting for each formal variable x_j the coordinate α_j. We denote this relationship symbolically by $X = V(S)$ where

$$V(S) := \{\alpha \in \mathbb{A}^n : f(\alpha) = 0 \text{ for all } f \in S\}$$

and in words by saying that

- X is the *affine variety cut out by the family of equations* $f = 0$ *for* $f \in S$,
- X is the *(affine) variety cut out by* S,
- X is the *locus of* S,

and perhaps with other turns of phrase that may evolve as we go.

3.2 Questions

There are many basic questions one can ask about such X. For instance, when (in terms of the defining equations f_α) is X nonempty? Can one algorithmically determine when that is the case? If so, can one "say something" about X? For instance, can one attach interesting and/or meaningful *(algebraic) invariants* to X? (Answer: yes, one can. Examples of such to be discussed later include dimension, degree, genus, function field, cohomology groups, etc.) Can one *classify* all X having prescribed invariants? When does one such X "look like" some other X', perhaps after throwing away "small" subsets? When the underlying field k has additional structure, such as a *metric* or *topology* (example: $k = \mathbb{C}$; interesting non-example: $k = \overline{\mathbf{F}_p}$, the algebraic closure of the finite field with p elements), how does that additional structure on k interact with the algebraic structure? For instance, how do *topological invariants* relate to algebraic ones? Can the former be described in terms of the latter? Can one find algebraic analogues over $\overline{\mathbf{F}_p}$ of topological invariants that make sense initially only over $\mathbb{C}$?

3.3 Why polynomials?

Why consider equations defined by *polynomials*, rather than (say) those defined by continuous, smooth, or analytic functions? For one, polynomials are intrinsically interesting. Another good reason is that over some fields, such as $\overline{\mathbf{F}_p}$, there is not initially much structure to play with other than that afforded by addition and multiplication. One is thus forced to work algebraically, i.e., with polynomials. The class of polynomials is substantially more *rigid* than other classes of functions (e.g., continuous functions). This rigidity is a two-sided coin. On the one hand, it imparts some *inflexibility.* For instance, there are no non-constant polynomials $f \in \mathbb{C}[x]$ in one variable satisfying the functional equation

$f(x+1) = f(x)$, while there are plenty of non-constant continuous functions with that property (e.g., $f(x) := e^{2\pi i x}$). On the other hand, polynomial functions are more *structured* than their topological and analytic counterparts, so one can often say more about them.

3.4 Special cases: linear algebra, Galois theory

There are special cases of the above questions which are somewhat more basic. For instance, the case that each f_i is a *linear* polynomial is just the theory of *linear algebra*. In that case, *Gaussian elimination* provides an effective algorithm to determine when $X \neq \emptyset$ (and even to parametrize the solution set), while the most interesting invariant is the dimension. Another interesting case is when the number of variables is $n = 1$, which corresponds to the study of solutions to a polynomial equation in one variable; this study basically amounts to *Galois theory*, in which the basic invariant is the degree of an algebraic number (or equivalently, of the field extension it generates). So algebraic geometry can be viewed as a common generalization of the two.

3.5 Why work over an (algebraically closed) field?

Why work over a field k? And why take that field to be algebraically closed? On the one hand, the theory becomes simpler, and one can say more. But it can be interesting to work over more general rings. For instance, polynomial equations over the integers $\mathbf{Z}$ can be reduced over each finite field $\mathbf{F}_p$ and then regarded over their algebraic closures $\overline{\mathbf{F}_p}$; one can then ask what the families of solution sets over such distinct algebraically closed fields have in common, if anything. As another example, the polynomial ring $k[x, y]$ in two variables x, y over the field k may often be profitably viewed as a polynomial ring $k[x][y]$ in one variable y over the ring $k[x]$.

3.6 Motivating problem: how to compute the genus algebraically?

Having finished a few motivational remarks, let us describe a motivating problem. Take $k := \mathbb{C}$ for now. The simplest affine variety not covered by the special cases mentioned above is the parabola $X := \{(x, y) \in \mathbb{C}^2 : y^2 = x\}$. We may view this as the total space of the branched cover of $\mathbb{C}^1$ obtained by the squaring map. That is, we may think of X as the glueing of two copies of $\mathbb{C}^1$ along the origin after cutting them along the line $(-\infty, 0)$ and identifying opposite edges of the top and bottom sheets. (I drew a picture in lecture of this and the following example that hopefully made some sense of them.) If one suitably compactifies X by adding a point at ∞, one obtains a copy of the Riemann sphere. For another example, consider $X := \{(x, y) \in \mathbb{C}^2 : y^2 = x(x-10)(x-20)(x-30)\}$. If we cut two copies of the complex plane along the intervals $(10, 20)$ and $(30, 40)$, glue them together along opposite edges, and identify the branch points $0, 10, 20, 30$ between the two copies, then we obtain X. If we compactify it by adding points at ∞ both to the top and bottom sheets, then we obtain a Riemann surface with "one hole," i.e., of topological genus one. One obtains similar pictures when X is the solution set to $y^2 = f(x)$ for higher-degree polynomials f; the number of holes (suitably counted in the presense of singularities arising in this case from repeated roots of f) defines a topological invariant known as the *genus* of X which is typically one less than the least integer exceeding half the degree of f. One can now ask: how can one describe such invariants algebraically? For instance, can one find an algebraic analogue of "number of holes" that makes sense even over fields like $\overline{\mathbf{F}_p}$?

4 Formal properties of the set-to-affine-variety function

4.1 Setup

Recall that an affine variety X is the set of solutions $\alpha \in k^n$ to a system of polynomial equations $f(\alpha) = 0$ for all f belonging to some collection of polynomials

$$S \subset k[x] := k[x_1, \dots, x_n],$$

i.e., $X = V(S)$ where

$$V(S) := \{a = (a_1, \dots, a_n) \in \mathbb{A}^n := k^n : f(a) = 0 \text{ for all } f \in S\}.$$

When S is a singleton $\{f\}$ for some $f \in k[x]$, we abbreviate $V(f) := V(\{f\})$. The "set-to-affine-variety" function

$$V : \{ \text{ subsets of } k[x] \ \} \to \{ \text{ affine } k\text{-varieties } \}$$

$$S \mapsto V(S)$$

satisfies some basic formal properties, which we now explicate. These properties are trivial in a technical sense, but important to master until they become second-nature.

4.2 Order-reversal

For one, the map V is *order-reversing* in the sense that[2]

$$S_1 \supset S_2 \implies V(S_1) \subset V(S_2). \tag{1}$$

In words, the above implication breaks down as follows:

- $S_1 \supset S_2$: "there are more equations $f = 0$ defined by $f \in S_1$ than by $f \in S_2$."

[2] We pause here to note that we adopt the convention that by $S \subset T$ (or equivalently $T \supset S$) we mean that every element of S is contained in T. We reserve the notation $S \subsetneq T$ to denote a strict containment. In some treatments, such relationships are instead denoted respectively by $S \subseteq T$ and $S \subset T$.

- $V(S_1) \subset V(S_2)$: "there are fewer solutions to the equations defined by S_1 than to those defined by S_2."
- $\implies$: "the more equations, the fewer solutions."

A particular example is worth singling out: if S is nonempty, and $f \in S$ is any element, then
$$V(S) \subset V(f).$$
In words, "any solution to a system of equations is, in particular, a solution to any equation in the system." We shall frequently use this containment as well as its complemented form $D(f) \subset \mathbb{A}^n - V(S)$, where
$$D(f) := \mathbb{A}^n - V(f) = \{\alpha \in \mathbb{A}^n : f(\alpha) \neq 0\}.$$

4.3 Extreme cases

On the one hand, the empty set of equations has as many solutions as possible:
$$V(\emptyset) = \mathbb{A}^n.$$
On the other hand, the universal set of equations contains in particular the contradictory equation $1 = 0$, thus
$$V(k[x]) = V(1) = \emptyset.$$

4.4 Boolean operations

We now consider boolean operations. The solution set to the union of several systems of equations is the intersection of the solution sets to the individual systems, that is to say, for any family $(S_i)_i$ of subsets of $k[x]$,
$$V(\cup_i S_i) = \cap_i V(S_i).$$
Since k is (a field, and hence in particular) an integral domain, a pair of equations $f_1(\alpha) = 0$ and $f_2(\alpha) = 0$ may always be combined into a single equation $f_1(\alpha)f_2(\alpha) = 0$, thus
$$V(f_1 f_2) = V(f_1) \cup V(f_2).$$
More generally, if S_1, S_2 are subsets of $k[x]$ and $S_1 S_2$ denotes the set of products $f_1 f_2$ with $f_1 \in S_1, f_2 \in S_2$, then
$$V(S_1 S_2) = V(S_1) \cup V(S_2).$$

4.5 Zariski topology

A particular consequence of the preceeding sections is that the collection of affine varieties inside $\mathbb{A}^n$ contains the empty set $\emptyset$, the entire space $\mathbb{A}^n$, and is closed under intersections and finite unions. In other words, this collection consists of the closed sets for a topology, called the *Zariski topology.* This topology has the property that the sets $V(f)$ for $f \in k[x]$ form a basis for the closed sets, or equivalently, their complements $D(f)$ form a basis for the open sets; we shall refer to the latter as *basic open subsets.* We spell this out a bit more explicitly:

- A closed set for the Zariski topology (by definition) is a *solution set* for a system of equations, that is to say, the set of all $\alpha \in \mathbb{A}^n$ where each equation $f(\alpha) = 0$ holds for f traversing some system $S \subset k[x]$.

- An open set is a *solution-failure set* for a system of equations, that is to say, the set of all $\alpha \in \mathbb{A}^n$ which for *some* f in the given system $S \subset k[x]$ fail to satisfy the equation $f(\alpha) = 0$, or equivalently, which *do* satisfy the *inequation* $f(\alpha) \neq 0$.

- The assertion that the $V(f)$ form a closed basis for the Zariski topology, i.e., that any closed set is an intersection of such, reads as follows: "Any system of equations can be defined as the intersection of individual equations."

- That the $D(f)$ form an open basis for the Zariski topology, i.e., that any open is a union of such, reads: "Any element of the solution-failure set for a system of equations fails to satisfy some equation in the system," or equivalently, "Any solution-failure set to a system of equations may be defined as the union of the solution sets to inequations."

Some examples should appear on the first homework and in the exercise sessions.

4.6 The vanishing ideal and the Zariski closure

Definition 2. The *(affine) vanishing ideal* $I(X)$ of a subset X of $\mathbb{A}^n$ is the set

$$I(X) := \{f \in k[x] : f|_X = 0\}$$

of polynomials that vanish on X.

Denote by $\mathrm{Zcl}(X)$ the Zariski closure of X. We have the following simple fact:

Exercise 1. For $X \subset \mathbb{A}^n$, one has $V(I(X)) = \mathrm{Zcl}(X)$. In particular, if X is an affine variety, then $V(I(X)) = X$.

Remark 3. In practice, one often computes the Zariski closure of a subset Y of $\mathbb{A}^n$ by guessing the answer, call it $Z \supset Y$, and then verifying that

$$f \in k[x], f|_Y = 0 \implies f|_Z = 0,$$

i.e., that each polynomial vanishing on Y also vanishes on Z. One then knows (because the $V(f)$ give a basis for the closed sets) that Z is the Zariski closure of Y. This is often simpler than explicitly computing the vanishing ideal $I(Y)$ and its locus $\mathrm{Zcl}(Y) = V(I(Y))$.

4.7 Ideal generation

We have $0 + 0 = 0$ and $c \cdot 0 = 0$ for all $c \in k$, so if $a_1, \ldots, a_m, f_1, \ldots, f_m \in k[x]$ and $\alpha \in \mathbb{A}^n$ satisfy

$$f_1(\alpha) = \cdots f_m(\alpha) = 0,$$

then also

$$a_1(\alpha)f_1(\alpha) + \cdots + a_m(\alpha)f_m(\alpha) = 0.$$

In particular, if S is any subset of $k[x]$, then since every element of the ideal (S) generated by S can be written as $a_1 f_1 + \cdots + a_m f_m$ for some $a_1, \ldots, a_m \in k[x]$ and some $f_1, \ldots, f_m \in S$, we see that

$$V(S) = V((S)).$$

In particular, *every affine variety in* $\mathbb{A}^n$ *is of the form* $V(\mathfrak{a})$ *for some ideal* $\mathfrak{a}$.

Since $k[x]$ is noetherian (the "Hilbert basis theorem"), each ideal (S) is finitely-generated, say by m elements $f_1, \ldots, f_m$; it follows that

$$V(S) = V(\{f_1, \ldots, f_m\}).$$

In other words, each affine variety is the solution set to a *finite* system of polynomial equations.

5 Algebra, geometry, and the Nullstellensatz

5.1 Motivating question: does the existence of solutions over some enormous superfield imply their existence over the base field?

Suppose one is given a (finite) system of polynomial equations in finitely-many variables over an algebraically closed field k, and that the system is known to admit a solution over some larger field $\Omega \supset k$. (Think of Ω as being "absolutely enormous," *much* larger than the original field, and thus a much more likely place, one could conceivably imagine, to find a solution.) Must the system then also have a solution over k? We will shortly be able to answer this question (see Section 5.6).

Note that if the system actually consists of *linear equations*, then even without the assumption $k = \overline{k}$, Gaussian elimination either returns a solution over the original field k or produces a contradiction of the shape "$1 = 0$" certifying the impossibility of a solution over any larger field.

On the other hand, if the system consists of a single polynomial equation $f(x_1) = 0$ in one variable with coefficients in k, then we know from field theory that if f is not a nonzero constant polynomial (in which case no solution can exist in any field) then there exists a solution in the algebraically closed field k. So the question we have raised is whether the affirmative answer in these simple cases extends to the general case.

5.2 Non-polynomial examples of the algebra-to-geometry correspondence

Quite generally, give some sort of geometric space X one obtains an algebraic structure by considering something like a ring A of functions on X, perhaps interpreted in a sufficiently liberal sense. Conversely, one can often study an algebraic structure A by realizing it as something like the space of functions on a suitable geometric space X. In this section we motivate the discussion to follow with some *non-polynomial* examples along such lines.

5.2.1 Continuous functions on a compact Hausdorff space

Let X be a compact Hausdorff topological space. Denote by $A := C(X)$ the ring of continuous functions $f : X \to \mathbb{C}$. For each $x \in X$, we obtain an evaluation map

$$\mathrm{eval}_x : A \to \mathbb{C}$$

given by $\mathrm{eval}_x(f) := f(x)$ and also a maximal ideal

$$\mathfrak{m}_x := \ker(\mathrm{eval}_x) \subset A$$

consisting of those functions vanishing at x. Denote by $\mathrm{Specm}(A)$ the set of maximal ideals $\mathfrak{m}$ in A equipped with the *Zariski topology*, for which a basis of open sets is given by the "*doesn't-vanish*" sets

$$D(f) := \{\mathfrak{m} : f \notin \mathfrak{m}\} \text{ for each } f \in A.$$

The terminology is motivated by noting that for each point $x \in X$, the maximal ideal $\mathfrak{m}_x$ belongs to the doesn't-vanish-set $D(f)$ if and only if f does not vanish at x, i.e., $f(x) \neq 0$. We then have the following fundamental fact:[3]

Theorem 4. *The natural map $X \to \mathrm{Specm}(A)$ given by $x \mapsto \mathfrak{m}_x$ is a topological isomorphism.*

The injectivity and topological assertions are not hard to verify using Urysohn's lemma and the definitions; the key step is to show that every maximal ideal of A arises from evaluation at some point, for the proof of which one argues as in the Stone–Weierstrauss theorem. The Nullstellensatz (to be discussed shortly) provides the analogue of this last step in the setting of polynomial equations over an algebraically closed field.

In summary, we have recorded an example where a geometric structure (the compact Hausdorff topological space) gives rise to an algebraic structure (the ring of continuous functions) from which the original geometry can in turn be recovered (by taking the set of maximal ideals equipped with the Zariski topology).

[3]This theorem would nowadays be regarded as a special case due to Gelfand–Kolmogorov of the commutative case of the Gelfand–Naimark theorem.

5.2.2 Commuting hermitian matrices

To give a simple example illustrating the algebra-to-geometry direction of the correspondence under consideration, consider a collection

$$A \subset \mathrm{Mat}_n(\mathbb{C})$$

of commuting hermitian matrices. (The case in which A is a singleton, or when it is the polynomial ring generated by a single element, is already worth considering.) We think of them as self-adjoint operators on $\mathbb{C}^n$. They may be simultaneously diagonalized with respect to some basis of eigenvectors. Collecting together those eigenvectors which belong to an individual eigenspace of each element of A, we obtain a decomposition (canonical up to permuting summands)

$$\mathbb{C}^n = E_1 \oplus \cdots \oplus E_k$$

where each $a \in A$ acts on each E_i via some scalar $\lambda_i(a) \in \mathbb{C}$ and so that the maps $\lambda_i : A \to \mathbb{C}$ are distinct. The set

$$X := \{\lambda_1, \dots, \lambda_k\}$$

is then referred to as the *spectrum* of A. We derive in this way an identification

$$A \hookrightarrow \mathrm{Func}(X, \mathbb{C})$$

of A with a family of functions on its spectrum X, with the operator $a \in A$ corresponding to the function

$$\hat{a} : X \to \mathbb{C}$$

defined by $\hat{a}(\lambda_i) := \lambda_i(a)$.

In summary, we have seen an example in which algebraic objects (commuting hermitian matrices) may be realized as functions on a geometric space (their spectrum).

Remark 5. Most of this example makes sense without the "hermitian" assumption if we define eigenvectors in a suitably generalized sense, but the correspondence becomes less perfect in that generality because of pairs of matrices like $\left(\begin{smallmatrix} z & 0 \\ 0 & z \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} z & 1 \\ 0 & z \end{smallmatrix}\right)$, which have the same spectrum but are non-similar. This illustrates the general principle that "nilpotent behavior" weakens (or perhaps enriches) the algebra-geometry correspondence.

Remark 6. One can expand (and unify) the above two motivating examples (cf. the spectral theory of C^*-algebras) but we do not pause to do so here. The point is just to emphasize that the general correspondence under consideration applies also in settings having nothing to do with polynomials.

5.3 Algebraic formulations of the Nullstellensatz

Theorem 7 (Nullstellensatz in the form of "Zariski's lemma")**.** *Let k be a field, and let A be a finite type k-algebra*[4] *which is also a field. Then A is a finite algebraic field extension of k.*

[4] "finite type k-algebra" is a synonym for "finitely-generated k-algebra." Apologies for any confusion if I inadvertently switch between the two.

Proof. The proof is rather elegantly spelled out in the hints to Exercise 5.18 of Atiyah–Macdonald; we do not reproduce it here. The case that A is generated as a k-algebra by a single element is one of the first lemmas in a typical Galois theory course. $\square$

Remark 8. The case that A is generated as a k-algebra by a single element is one of the first lemmas in a typical Galois theory course.

We spell out some basic consequences, of which we shall only use Corollary 12 and Corollary 14. Most of the assertions to follow are steps involved in solutions to various exercises in Chapter 5 of Atiyah–Macdonald. We did not cover any of these in lecture, and so have included them here for completeness. In all of these, k is be assumed to be a field.

Corollary 9. *Let A be a finite type k-algebra. Then each maximal ideal $\mathfrak{m}$ of A arises as the kernel of some surjective k-algebra map $\phi : A \to E$, where E is a finite field extension of k.*

Proof. Apply Theorem 7 to the finite type k-algebra $A/\mathfrak{m}$, define $E := A/\mathfrak{m}$ and let $\phi : A \to E = A/\mathfrak{m}$ be the canonical projection. $\square$

Corollary 10. *Let $f : A \to B$ be a map of k-algebras with B of finite type. Then for each maximal ideal $\mathfrak{n}$ of B, the preimage $\mathfrak{m} := f^{-1}(\mathfrak{n})$ is a maximal ideal of A.*

Proof. By the previous corollary, $\mathfrak{n}$ is the kernel of a map $\phi : B \to E$ for some finite extension E of k. Its preimage $\mathfrak{m} := f^{-1}(\mathfrak{n})$ is thus the kernel of the composition $\phi \circ f : A \to E$. The image $(\phi \circ f)(A) \cong A/\mathfrak{m}$ is a ring contained between the field k and its finite extension E, hence is a field, hence $\mathfrak{m}$ is maximal, as claimed. $\square$

Corollary 11. *Let A be a finite type k-algebra. Each prime ideal $\mathfrak{p}$ of A is the intersection of the maximal ideals $\mathfrak{m}$ containing it:*

$$\mathfrak{p} = \cap_{\mathfrak{m} \supset \mathfrak{p}} \mathfrak{m}.$$

Proof. By replacing A with $A/\mathfrak{p}$, we reduce to showing that if A is an integral domain and $f \in A$ is a nonzero element, then there exists a maximal ideal $\mathfrak{m}$ of A with $f \notin \mathfrak{m}$. Our hypotheses imply that the localization A_f is not the zero ring, and so has some maximal ideal $\mathfrak{n}$. Since A_f is a finite type k-algebra, the previous corollary implies that the preimage $\mathfrak{m} := S_f^{-1}(\mathfrak{n})$ under $S_f : A \to A_f$ is a maximal ideal of A, which by basic properties of localization does not contain f, as required. $\square$

Corollary 12. *Let A be a finite type k-algebra. Then the intersection of all maximal ideals in A is the nilradical of A:*

$$\cap_{\mathfrak{m} \subset A} = \mathfrak{N}(A).$$

Proof. Indeed, $\mathfrak{N}(A) = \cap_{\mathfrak{p} \subset A} \mathfrak{p}$ with the intersection taken over prime ideals, which by the previous corollary coincides with the intersection taken over maximal ideals. $\square$

Corollary 13. *Let A be a k-algebra. Then the natural map*

$$\mathrm{Hom}_k(A, k) \to \mathrm{Specm}(A)$$

$$\phi \mapsto \ker(\phi)$$

is injective.

Proof. (Recall that in lecture, we gave a concrete proof of this fact where it was needed in the geometric setting.) Let $\phi_1, \phi_2 \in \mathrm{Hom}_k(A, k)$ with $\ker(\phi_1) = \ker(\phi_2)$. Since in particular $\ker(\phi_1) \subset \ker(\phi_2)$, there exists $\tau \in \mathrm{Hom}_k(k, k)$ so that $\phi_2 = \tau \circ \phi_1$. Since $\mathrm{Hom}_k(k, k) = \{1\}$, we have $\tau = 1$ and so $\phi_2 = \phi_1$. $\square$

Corollary 14. *Let k be an algebraically closed field, and let A be a finite type k-algebra. Then the natural map*

$$\mathrm{Hom}_k(A, k) \to \mathrm{Specm}(A)$$

$$\phi \mapsto \ker(\phi)$$

is a bijection.

Proof. By the previous corollary, we need only verify the surjectivity. Thus, let $\mathfrak{m}$ be a maximal ideal of A. By Corollary 9, $\mathfrak{m}$ is the kernel of some map $\phi : A \to E$, where E is a finite field extension of k. Since k is algebraically closed, we must have $E = k$. Therefore $\mathfrak{m} = \ker(\phi)$ for some $\phi \in \mathrm{Hom}_k(A, k)$, as required. $\square$

5.4 Points, morphisms, and maximal ideals

Let $X \subset \mathbb{A}^n$ be an affine k-variety, where k is assumed here and henceforth to be *algebraically closed*. By the discussion of Section 4, it arises as the locus

$$X = V(\mathfrak{a})$$

of some *ideal*

$$\mathfrak{a} \subset k[x] := k[x_1, \ldots, x_n].$$

Denote by

$$A := k[x]/\mathfrak{a}$$

the quotient ring. This ring is generated by the images $\overline{x}_1, \ldots, \overline{x}_n$ in A of the coordinate functions $x_1, \ldots, x_n \in k[x]$.

Let $a \in A$ and $\alpha \in X$. Since every element of $\mathfrak{a}$ vanishes identically on X, the *value* of a at α, denoted $a(\alpha)$, is well-defined.[5] Write

$$a|_X : X \to k$$

[5] Note that it does not in general make sense to evaluate $a \in A$ at an arbitrary point $\alpha \in \mathbb{A}^n$.

for the function on X so obtained. For each $\alpha \in X$, we have the *evaluation* map

$$\mathrm{eval}_\alpha \in \mathrm{Hom}_k(A, k)$$

given by $\mathrm{eval}_\alpha(a) := a(\alpha)$. Its kernel

$$\mathfrak{m}_\alpha := \ker(\mathrm{eval}_\alpha)$$

is the maximal ideal consisting of all $a \in A$ for which $a(\alpha) = 0$. Conversely, given $\phi \in \mathrm{Hom}_k(A, k)$, we may define a point[6]

$$\mathrm{pt}(\phi) := (\phi(\overline{x}_1), \dots, \phi(\overline{x}_n)) \in X.$$

Theorem 15. *The natural maps*

$$X \to \mathrm{Hom}_k(A, k) \to \mathrm{Specm}(A)$$

$$\alpha \mapsto \mathrm{eval}_\alpha \mapsto \mathfrak{m}_\alpha$$

$$\mathrm{pt}(\phi) \leftarrow\!\shortmid \phi \mapsto \ker(\phi)$$

are mutually inverse bijections. Moreover, the kernel $\{a \in A : a|_X = 0\}$ *of the map* $a \mapsto a|_X$ *is the nilradical* $\mathfrak{N}(A)$ *of* A*. Finally, the bijection* $X \to \mathrm{Specm}(A)$ *is a homeomorphism with respect to the Zariski topologies.*

Proof. We verify first that the associations $\alpha \mapsto \mathrm{eval}_\alpha$ and $\phi \mapsto \mathrm{pt}(\phi)$ are mutually inverse. Indeed,

$$\mathrm{pt}(\mathrm{eval}_\alpha) = (\mathrm{eval}_\alpha(\overline{x}_1), \dots, \mathrm{eval}_\alpha(\overline{x}_n)) = (\alpha_1, \dots, \alpha_n) = \alpha,$$

while for each of the generators $\overline{x}_j$ of A,

$$\mathrm{eval}_{\mathrm{pt}(\phi)}(\overline{x}_j) = \overline{x}_j(\mathrm{pt}(\phi)) = \phi(\overline{x}_j).$$

That the second map is a bijection follows from Corollary 14. For the penultimate assertion, we have

$$\{a \in A : a|_X = 0\} = \cap_{\alpha \in X} \mathfrak{m}_\alpha$$

because a function on a set vanishes iff it vanishes at each point. Since $\alpha \mapsto \mathfrak{m}_\alpha$ is onto,

$$\cap_{\alpha \in X} \mathfrak{m}_\alpha = \cap_{\mathfrak{m} \subset A} \mathfrak{m}$$

with the intersection taken over all maximal ideals. Corollary 12 then tells us that $\cap_{\mathfrak{m} \subset A} \mathfrak{m} = \mathfrak{N}(A)$. Regarding the topologies, the bases of opens are given by $D(f) = \{\alpha \in X : f(\alpha) \neq 0\} \subset X$ for $f \in k[x]$ and by $D(a) = \{\mathfrak{m} \in \mathrm{Specm}(A) : a \notin \mathfrak{m}\}$ for $a \in A$. Since $f(\alpha)$ is nonzero iff its image in A does not belong to $\mathfrak{m}_\alpha$, the map $\alpha \mapsto \mathfrak{m}_\alpha$ is indeed a homeomorphism. □

Remark 16. The proof of the first bijection $X \to \mathrm{Hom}_k(A, k)$ in Theorem 15 did not make use of the assumption that k is algebraically closed.

[6] This point belongs to X because each $f \in \mathfrak{a}$ satisfies $f(\mathrm{pt}(\phi)) = \phi(f \mod \mathfrak{a}) = 0$.

5.5 The affine coordinate ring and vanishing ideal

(We continue to assume that k is algebraically closed.) Let $X \subset \mathbb{A}^n$ be an affine variety, say $X = V(\mathfrak{a})$ with $\mathfrak{a} \subset k[x] := k[x_1, \dots, x_n]$ an ideal, as usual.

Definition 17. The *affine coordinate ring* $A(X)$ of X is the image of $k[x]$ under the restriction map to the space of functions on X, that is to say,

$$A(X) := \{f|_X : X \to k \mid f \in k[x]\}.$$

Element $a : X \to k$ of the affine coordinate ring are called *polynomial functions* on X.

Recall that the *(affine) vanishing ideal* of X is the kernel of this restriction map, i.e.,

$$I(X) := \ker(k[x] \twoheadrightarrow A(X)) = \{f \in k[x] : f|_X = 0\}.$$

Thus restriction induces an identification

$$k[x]/I(X) = A(X),$$

which in some treatments is taken as the definition of $A(X)$. Clearly

$$\mathfrak{a} \subset I(X),$$

since each polynomial f contributing a defining equation $f = 0$ for X must, in particular, vanish on X. Moreover, if we write

$$A := k[x]/\mathfrak{a}$$

as before, then the restriction map $k[x] \twoheadrightarrow A(X)$ factors as

$$k[x] \twoheadrightarrow A \twoheadrightarrow A(X)$$

where $A \mapsto A(X)$ is the map $f \mapsto f|_X$ discussed above in Section 5.4.

Remark 18.

1. Let A be a ring. Define $A^{\mathrm{red}} := A/\mathfrak{N}(A)$. Say that a ring A is *reduced* if $A^{red} = A$, i.e., if $\mathfrak{N}(A) = 0$, that is to say, if A has no nonzero nilpotents.

2. Let X be an affine variety. Then the affine coordinate ring $A := A(X)$ is of finite type and reduced. It is given by $A = k[x]/\mathfrak{a}$ with $\mathfrak{a} := I(X)$, which by Exercise 1 satisfies $V(\mathfrak{a}) = X$.

3. Let A be any finite type k-algebra. Then there exists a polynomial ring $k[x] := k[x_1, \dots, x_n]$ which surjects onto A. Denote by $\mathfrak{a}$ the kernel of some such surjection, so that we may identify $A = k[x]/\mathfrak{a}$. Theorem 15 then identifies $\operatorname{Specm}(A)$ with the affine variety $X = V(\mathfrak{a})$ and tells us that the kernel of the natural map $A \twoheadrightarrow A(X)$ is the nilradical $\mathfrak{N}(A)$, so that $A^{\mathrm{red}} \cong A(X)$.

4. Suppose now that A is a finite type reduced k-algebra, and write $A = k[x]/\mathfrak{a}$ as before; the ideal $\mathfrak{a}$ is then necessarily *radical*, i.e., $r(\mathfrak{a}) = \mathfrak{a}$, and the natural map $f \mapsto f|_X$ induces an isomorphism $A \cong A(X)$. Therefore every finite type reduced k-algebra may be realized as the affine coordinate ring of some affine variety X, which is in all cases topologically isomorphic to $\operatorname{Specm}(A)$. The choice of X, that is to say, the realization of $\operatorname{Specm}(A)$ in some affine space $\mathbb{A}^n$, corresponds to the choice of surjection $k[x_1, \ldots, x_n] \to A$.

5.6 Geometric interpretations of the Nullstellensatz

(We continue to assume that k is algebraically closed and shall forget.) In this section $k[x] := k[x_1, \ldots, x_n]$ and $\mathfrak{a} \subset k[x]$ denotes an ideal.

5.6.1 The weak form: the only obstruction to solving simultaneous equations is the obvious one

Corollary 19. $V(\mathfrak{a}) = \emptyset$ *iff* $\mathfrak{a} = (1) := k[x]$.

Proof. Write $A := k[x]/\mathfrak{a}$. By Theorem 15, each of the following is evidently equivalent to the next:

- $V(\mathfrak{a}) = \emptyset$
- $\operatorname{Specm}(A) = \emptyset$
- $A = 0$
- $\mathfrak{a} = (1)$.

We have established the required equivalence. □

The interpretation of the above result is as follows: Suppose one wishes to find a solution $\alpha \in k^n$ to some system of polynomial equations

$$f_1(\alpha) = \cdots = f_m(\alpha) = 0 \tag{2}$$

with each $f_i \in k[x]$, as usual. An obvious *obstruction* to finding a solution is the existence of an identity in $k[x]$ of the shape

$$a_1 f_1 + \cdots + a_m f_m = 1 \text{ for some } a_1, \ldots, a_m \in k[x], \tag{3}$$

because substituting a putative solution α would then produce the contradiction $0 = 1$. Corollary 19 says that the existence of an identity of the shape (3) is in fact the *only* obstruction. Indeed, let $\mathfrak{a} := (f_1, \ldots, f_m)$. Then each of the following assertions is equivalent to the next one:

- There is no solution to the system (2).
- $V(\{f_1, \ldots, f_m\}) = \emptyset$.

- $V(\mathfrak{a}) = \emptyset$.
- $\mathfrak{a} = (1)$. (Here we have crucially used Corollary 19.)
- There exists an identity of the shape (3).

Exercise 2. Determine the answer to the motivating question from Section 5.1.

5.6.2 The strong form: the only obstruction to solving simultaneous equations and inequations is the obvious one

Corollary 20. $I(V(\mathfrak{a})) = r(\mathfrak{a})$.

Proof. Write $A := k[x]/\mathfrak{a}$ and $X := V(\mathfrak{a})$. By the above discussion of Section 5.5, $I(X)$ is the the preimage under $\pi : k[x] \to A$ of $\{f \in A : f|_X = 0\}$. By Theorem 15, the latter is the nilradical $\mathfrak{N}(A)$, whose preimage is $\pi^{-1}(\mathfrak{N}(A)) = r(\mathfrak{a})$, as claimed. $\square$

To interpret Corollary 20, suppose one wishes to find a solution $\alpha \in k^n$ to some system of polynomial equations and inequations

$$f_1(\alpha) = \cdots = f_m(\alpha) = 0, \quad g_1(\alpha) \neq 0, \ldots, g_k(\alpha) \neq 0$$

for some $f_i, g_j \in k[x]$. Writing $g := g_1 \cdots g_k$ for the product of the g_j, this system is equivalent to the slightly simpler system

$$f_1(\alpha) = \cdots = f_m(\alpha) = 0, \quad g(\alpha) \neq 0 \tag{4}$$

upon which we henceforth focus attention. An obvious *obstruction* to finding a solution is the existence of an identity in $k[x]$ of the shape

$$a_1 f_1 + \cdots + a_m f_m = g^N \text{ for some } a_1, \ldots, a_m \in k[x], N \in \mathbf{Z}_{\geq 0}, \tag{5}$$

for then a solution would yield the contradiction $0 = g(\alpha)^N$ and yet $g(\alpha) \neq 0$. Corollary 20 says that the existence of an identity of the shape (5) is the only obstruction to solving the system (4). Indeed, writing $\mathfrak{a} := (f_1, \ldots, f_m)$ as before, each of the following assertions is equivalent to the next:

- There is no solution to the system (4).
- $f_1(\alpha) = \cdots = f_m(\alpha) = 0$ implies $g(\alpha) = 0$ for all $\alpha \in k^n$.
- $g(\alpha) = 0$ for all $\alpha \in V(\{f_1, \ldots, f_m\})$.
- $g(\alpha) = 0$ for all $\alpha \in V(\mathfrak{a})$.
- $g|_{V(\mathfrak{a})} = 0$, i.e., $g \in I(V(\mathfrak{a}))$.
- $g \in r(\mathfrak{a})$. (Here we have used Corollary 20.)
- There exists an identity of the shape (5).

Remark 21. One can deduce the apparently stronger Corollary 20 from the apparently weaker Corollary 19 by the "trick of Rabinowitsch," as follows: the system (4) has a solution iff the system

$$f_1(\alpha) = \cdots = f_m(\alpha) = 1 - \beta g(\alpha) = 0 \tag{6}$$

has a solution in the variables $\alpha_1, \ldots, \alpha_n, \beta \in k$, and the ideal $(f_1, \ldots, f_m, 1-yg)$ in the ring $k[x_1, \ldots, x_n, y]$ is the unit ideal iff (plugging in $y := 1/g$ and clearing denominators) some power of g belongs to $(f_1, \ldots, f_m)$. In the development recorded here, this trick corresponds to the localization step in the proof of Corollary 11. See also Section 6.10.

5.6.3 When do two systems of equations have the same sets of solutions?

There exist distinct systems of equations which have the same solution set. For instance, the locus of a polynomial $f \in k[x]$ is the same as that of f^2. One interpretation of Corollary 20 is that *this obstruction is the only one*, in the following precise sense:

Corollary 22. *Let $\mathfrak{a}_1, \mathfrak{a}_2 \subset k[x]$ be two ideals for which*

$$V(\mathfrak{a}_1) = V(\mathfrak{a}_2)$$

and

$$f^2 \in \mathfrak{a}_j \implies f \in \mathfrak{a}_j \text{ for } j = 1, 2 \text{ and } f \in k[x].$$

Then $\mathfrak{a}_1 = \mathfrak{a}_2$.

Proof. Our hypotheses may be seen to imply that $r(\mathfrak{a}_1) = \mathfrak{a}_1$ and $r(\mathfrak{a}_2) = \mathfrak{a}_2$. By Corollary 20, it follows that

$$\mathfrak{a}_1 = I(V(\mathfrak{a}_1)) = I(V(\mathfrak{a}_2)) = \mathfrak{a}_2.$$

□

5.7 Irreducibility versus primality

Definition 23. Let us say that a topological space X is *irreducible* if any of the following equivalent conditions hold:

- Every open subset of X is dense.
- Every pair of nonempty open subsets of X have nonempty intersection.
- Every pair of proper closed subsets Z_1, Z_2 of X have proper union, that is,
$$Z_1 \subsetneq X, Z_2 \subsetneq X \implies Z_1 \cup Z_2 \subsetneq X.$$
- No nonempty open subset of X is contained in a proper closed subset of X.

A subset of a topological space will be called irreducible if it is irreducible with respect to the induced topology.

Since affine varieties are, in particular, in topological spaces, it makes sense to speak of an affine variety (or any subset thereof) being irreducible.

Exercise 3. Let $k[x] := k[x_1, \ldots, x_n]$. If $\mathfrak{p} \subset k[x]$ is a prime ideal, then $V(\mathfrak{p})$ is irreducible. If $X \subset \mathbb{A}^n$ is an irreducible affine variety, then $I(X)$ is a prime ideal.

Combining with the results from the previous section, one obtains

Corollary 24. *Let $k[x] := k[x_1, \ldots, x_n]$ The map V induces natural order-reversing bijections*

$$\{ \text{ radical ideals } \mathfrak{a} \subset k[x] \ \} \to \{ \text{ affine } k\text{-varieties } X \subset \mathbb{A}^n \ \}$$

and

$$\{ \text{ prime ideals } \mathfrak{p} \subset k[x] \ \} \to \{ \text{ irreducible affine } k\text{-varieties } X \subset \mathbb{A}^n \ \}$$

and

$$\{ \text{ maximal ideals } \mathfrak{m} \subset k[x] \ \} \to \{ \text{ points } a = (a_1, \ldots, a_n) \in \mathbb{A}^n \ \}.$$

6 Polynomial maps between affine varieties

6.1 Motivating question: how to interpret geometrically the basic classes of ring morphisms?

We have seen a correspondence between certain geometric structures, namely affine k-varieties X, and certain algebraic structures, namely their affine coordinate rings $A := A(X)$, which are finite type reduced k-algebras. The aim of this section is to begin studying the geometric incarnation of morphisms between the algebraic objects, that is to say, k-algebras morphisms $A \to B$. Besides developing further our foundations, we will discuss what it means (and what it does not mean) geometrically for the map $A \to B$ to be

- a localization map $A \to A_a$ $(a \in A)$,
- surjective, or equivalently up to isomorphism, a quotient map $A \to A/\mathfrak{a}$,
- injective, or
- an isomorphism.

6.2 Setting

Before diving in, let us briefly recap. Let $k[x] := k[x_1, \ldots, x_n]$ and let $X \subset \mathbb{A}^n = \operatorname{Specm}(k[x])$ be an affine variety with affine coordinate ring $A := A(X)$ generated by the images $\overline{x}_1, \ldots, \overline{x}_n$ of the coordinate functions $x_1, \ldots, x_n$; it is a finite type reduced k-algebra of the form $A = k[x]/I(X)$ for some radical ideal $I(X)$ for which $X = V(I(X))$, and may (by its definition) be regarded as the subspace

$$A \subset \operatorname{Func}(X, k)$$

of polynomial functions $a : X \to k$ inside the larger space $\operatorname{Func}(X, k)$ of all (possibly uninteresting) functions $X \to k$. We saw last time that there are identifications

$$X \simeq \operatorname{Hom}_k(A, k) \simeq \operatorname{Specm}(A)$$

$$\alpha \mapsto \operatorname{eval}_\alpha \mapsto \mathfrak{m}_\alpha$$

$$(\phi(\overline{x}_1), \ldots, \phi(\overline{x}_n)) =: \operatorname{pt}(\phi) \mapsfrom \phi \mapsto \ker(\phi)$$

with the first map given by sending a point to its corresponding evaluation morphism $\operatorname{eval}_\alpha := [f \mapsto f(\alpha)]$ with inverse recovering the point by evaluating at the coordinate functions, the second map by taking kernels, and the composition by taking the kernel of evaluation, i.e., by attaching to a point the maximal ideal consisting of those functions that vanish at it.

We now set $k[y] := k[y_1, \ldots, y_m]$ and consider a second affine variety $Y \subset \mathbb{A}^m := \operatorname{Specm}(k[y])$ with affine coordinate ring $B := A(Y)$, so that similarly

$$Y \simeq \operatorname{Hom}_k(B, k) \simeq \operatorname{Specm}(B),$$

$$B \subset \operatorname{Func}(Y, k).$$

These notations and assumptions concerning the quadruple (X, A, Y, B) will be in force throughout the following sections.

6.3 Definition and algebraic characterization

Definition 25. A function $f : Y \to X$ between affine varieties will be called a *polynomial map* if for each polynomial function $a : X \to k$, that is to say, for each $a \in A$, the pullback $a \circ f : Y \to k$ is a polynomial function, that is to say, belongs to B. The set of such f will be denoted $\operatorname{Hom}_k(Y, X)$.

Thus polynomial maps $f : Y \to X$ of affine varieties are precisely those functions that induce a map

$$f^\sharp : A \to B$$

$$f^\sharp(a) := a \circ f$$

of affine coordinate rings via pullback. Note that for $\beta \in Y$, we may evaluate a function on X at the point $f(\beta)$ by first pulling the function back under f and then evaluating at β, i.e.,

$$\operatorname{eval}_{f(\beta)} = \operatorname{eval}_\beta \circ f^\sharp. \tag{7}$$

Exercise 4. $f^\sharp$ is a k-algebra morphism.

Theorem 26. *For affine varieties X, Y with affine coordinate rings A, B, the maps*

$$\mathrm{Hom}_k(Y, X) \to \mathrm{Hom}_k(A, B)$$
$$f \mapsto f^\sharp := [a \mapsto a \circ f]$$
$$(\phi(\overline{x}_1), \dots, \phi(\overline{x}_n)) =: \phi^\flat \leftarrow\!\shortmid \phi$$

are mutually inverse bijections.

Proof. Let $\beta \in Y$. Observe first that

$$\phi^\flat(\beta) = (\phi(\overline{x}_1)(\beta), \dots, \phi(\overline{x}_n)(\beta)) = \mathrm{pt}(\mathrm{eval}_\beta \circ \phi).$$

By (7),

$$(f^\sharp)^\flat(\beta) = \mathrm{pt}(\mathrm{eval}_\beta \circ f^\sharp) = \mathrm{pt}(\mathrm{eval}_{f(\beta)}) = f(\beta).$$

It remains only to verify that $\phi^\flat(A) \subset B$ and that $(\phi^\flat)^\sharp = \phi$: indeed, for $a \in A$,

$$(a \circ \phi^\flat)(\beta) = a(\mathrm{pt}(\mathrm{eval}_\beta \circ \phi)) = (\mathrm{eval}_\beta \circ \phi)(a) = \phi(a)(\beta),$$

whence $a \circ \phi^\flat = \phi(a) \in B$ and so $(\phi^\flat)^\sharp(a) = \phi(a)$. $\square$

Remark 27. Suppose $f : Y \to X$ is the restriction of some function $\mathbb{A}^m \to \mathbb{A}^n$ given in coordinates by polynomials, i.e., that there exist $\psi_1, \dots, \psi_n \in k[y]$ so that

$$f(\beta) = (\psi_1(\beta), \dots, \psi_n(\beta))$$

for all $\beta \in Y$. Then the pullback under f of a polynomial is a polynomial. Since polynomial functions are just restriction of polynomials, it follows that f defines a polynomial map. Conversely, every polynomial map f arises in this way upon taking for $\psi_i \in k[y]$ any representative for the image $f^\#(\overline{x}_i) \in B$ of the corresponding coordinate function on X.

Exercise 5. In Definition 25, it suffices to check that $a \circ f \in B$ for all a belonging to a set of *generators* for A, such as the set $\{\overline{x}_1, \dots, \overline{x}_n\}$ of coordinate functions.

Exercise 6. The identity $1_X : X \to X$ is a polynomial map. The composition $f \circ g : Z \to X$ of polynomial maps $f : Y \to X$ and $g : Z \to Y$ between affine varieties X, Y, Z with affine coordinate rings A, B, C is a polynomial map, and composition is associative, so affine varieties form a category. Moreover,

$$(f \circ g)^\sharp = g^\sharp \circ f^\sharp$$

as polynomial maps $A \to C$.

Definition 28. A polynomial map $f : Y \to X$ between affine varieties will be called an *isomorphism* if there exists a polynomial map $g : X \to Y$ so that $g \circ f = 1_X$ and $f \circ g = 1_Y$.

Remark 29. It follows that the maps $X \mapsto A(X)$ and $f \mapsto f^\sharp$ define a contravariant functor inducing an antiequivalence from the category of affine k-varieties (with polynomial maps) to the category of finite type reduced k-algebras, and similarly between the category of irreducible affine k-varieties and the category of finite type k-domains.[7]

6.4 Essential examples

Each of these examples deserves very careful study, with pictures (which should appear in lecture).

Exercise 7. For each of the following examples, verify by hand that $(f^\sharp)^\flat = f$.

Example 30 (Isomorphism between affine varieties with different embeddings in affine space)**.** Let

$$X := \mathbb{A}^1 := \operatorname{Specm} k[x_1],$$
$$Y := V(y_2 - y_1^2) \subset \mathbb{A}^2 := \operatorname{Specm} k[y_1, y_2],$$

then

$$Y \xrightarrow{f} X$$
$$(\beta_1, \beta_2) \mapsto (\beta_1)$$

is the downward projection from the standard parabola to the horizontal axis, corresponding to

$$A = k[x_1] \xrightarrow{f^\sharp} B = k[y_1, y_2]/(y_2 - y_1^2)$$
$$x_1 \mapsto y_1.$$

The maps f and $f^\sharp$ are isomorphisms between affine varieties X, Y which are embedded quite differently into affine space (see Section 6.5).

Example 31 (Bijective polynomial map which is not an isomorphism)**.** Let

$$X := V(x_1^2 - x_2^3) \subset \mathbb{A}^2 := \operatorname{Specm} k[x_1, x_2]$$

and

$$Y := \mathbb{A}^1 := \operatorname{Specm} k[y_1].$$

Then

$$Y \xrightarrow{f} X$$
$$(\beta_1) \mapsto (\beta_1^3, \beta_1^2)$$

is a polynomial map corresponding to

$$A = \frac{k[x_1, x_2]}{(x_1^2 - x_2^3)} \xrightarrow{f^\sharp} B = k[y_1]$$

[7] By a k-domain we mean a k-algebra which is an integral domain.

$$x_1 \mapsto y_1^3$$
$$x_2 \mapsto y_1^2.$$

The polynomial map f is bijective. Since $f^\sharp$ is not an isomorphism, f is likewise not an isomorphism.

Example 32 (Frobenius)**.** Suppose that k has characteristic $p > 0$. Then $\gamma \mapsto \gamma^p$ defines a field automorphism of k. Let

$$X := \mathbb{A}^1 := \operatorname{Specm} k[x_1],$$

$$Y := \mathbb{A}^1 := \operatorname{Specm} k[y_1],$$

and

$$Y \xrightarrow{f} X$$
$$(\beta_1) \mapsto (\beta_1^p),$$

corresponding to

$$A = k[x_1] \xrightarrow{f^\sharp} B = k[y_1]$$
$$x_1 \mapsto y_1^p.$$

Then f is bijective, but not an isomorphism.

Example 33 (Localization away from the zero set of a polynomial function)**.** Let

$$X := \mathbb{A}^1 := \operatorname{Specm} k[x_1],$$

so that $A = k[x_1]$. Fix $\gamma \in k$ and set $a := x_1 - \gamma \in A$. Let

$$Y := V(1 - az) = \{(\alpha, \beta) \in k^2 : \beta = \frac{1}{\alpha - \gamma}\} \subset \mathbb{A}^2 := \operatorname{Specm} k[x_1, z].$$

Then

$$Y \xrightarrow{f} X$$
$$(\beta_1, \beta_2) \mapsto (\beta_1)$$

is the downward projection of a hyperbola with vertical asymptote at γ, corresponding to the localization map

$$A = k[x_1] \xrightarrow{f^\sharp} B = \frac{k[x_1, z]}{(1 - az)} \cong k[x_1, \frac{1}{a}] \cong k[x_1]_a = A_a,$$
$$x_1 \mapsto x_1.$$

The polynomial map f is a homeomorphism onto its image $\mathbb{A}^1 - \{\gamma\} = X - V(a) = D(a)$.

Example 34 (Closed subvarieties)**.** Let $Y, X \subset \mathbb{A}^n := \operatorname{Specm} k[x]$ with $k[x] := k[x_1, \dots, x_n]$ be two affine varieties with $Y \subset X$. The inclusion map $\iota : Y \hookrightarrow X$ is a polynomial map corresponding to the surjective restriction map $\iota^\sharp : B \twoheadrightarrow A$ on coordinate rings.

Example 35 (The closed embedding of a point)**.** Let

$$X := \mathbb{A}^1 = \operatorname{Specm} k[x_1]$$

and

$$Y := \{0\} = \mathbb{A}^0 = \operatorname{Specm} k.$$

Fix $\gamma \in k$. Then

$$Y \xrightarrow{f} X$$

$$(0) \mapsto (\gamma)$$

corresponds to

$$A = k[x_1] \xrightarrow{f^\sharp} B = k$$

$$x_1 \mapsto \gamma.$$

The map $f^\sharp$ is surjective and f is a homeomorphism onto its closed image $\{\gamma\} = V(\ker f^\sharp)$.

Example 36 (The closed embedding of a parabola)**.** Let

$$X := \mathbb{A}^2 = \operatorname{Specm} k[x_1, x_2],$$

$$Y := V(y_2 - y_1^2) \subset \mathbb{A}^2 := \operatorname{Specm} k[y_1, y_2],$$

and

$$Y \xrightarrow{f} X,$$

$$(\beta_1, \beta_2) \to (\beta_1, \beta_2).$$

Then

$$A = k[x_1, x_2] \xrightarrow{f^\sharp} B = k[y_1, y_2]/(y_2 - y_1^2)$$

$$x_1 \mapsto y_1$$

$$x_2 \mapsto y_2.$$

The map $f^\sharp$ is surjective, and the polynomial map f is a homeomorphism onto the closed subset $V(\ker f^\sharp)$ of its codomain.

Example 37 (The inclusion of an affine variety into affine space)**.** Let $X \subset \mathbb{A}^n := \operatorname{Specm} k[x]$ with $k[x] := k[x_1, \ldots, x_n]$ be an affine variety. Then the inclusion map $X \hookrightarrow \mathbb{A}^n$ is a polynomial map corresponding to the surjective restriction map $k[x] \twoheadrightarrow A(X)$ on coordinate rings.

Example 38 (An open embedding which is not dense)**.** Let

$$X := V(x_1 x_2) \subset \mathbb{A}^2 := \operatorname{Specm} k[x_1, x_2],$$

$$Y := \mathbb{A}^1 := \operatorname{Specm} k[y_1],$$

then

$$Y \xrightarrow{f} X$$

$$(\beta_1) \mapsto (\beta_1, 0)$$

is the inclusion of the horizontal axis into the union of the horizontal and vertical axes, corresponding to

$$A = \frac{k[x_1, x_2]}{(x_1 x_2)} \xrightarrow{f^\sharp} B = k[y_1]$$

$$x_1 \mapsto y_1$$

$$x_2 \mapsto 0.$$

6.5 Isomorphism versus equality

Having equipped affine varieties with the structure of a category, we obtain as usual the notion of an *isomorphism* of affine varieties, namely a polynomial map with a two-sided inverse (see Definition 28). Theorem 26 tells us that two affine varieties are isomorphic if and only if their affine coordinate rings are isomorphic as k-algebras. However, this does *not* necessarily mean that they should be regarded as the "same variety:" they may be embedded in different affine spaces, or embedded in different ways in the same affine space, and some of their salient properties may consequently differ. For instance, although the line and parabola described in Example 30 are isomorphic, it would be wrong to regard them as being "the same;" for instance, the two obviously have different *degrees* in a sense to be defined precisely later. As a matter of terminology, one says that a property of X is

- *intrinsic* if it depends only on the isomorphism class of X, and
- *extrinsic* if it depends upon how X is embedded into affine space $\mathbb{A}^n$, or equivalently, upon the choice of surjection $k[x_1, \ldots, x_n] \twoheadrightarrow A(X)$.

For instance, the dimension of an affine variety will turn out to be intrinsic, Example 30 shows that the degree of a variety (to be defined) is extrinsic, and one interpretation of Theorem 26 is that *the intrinsic properties of an affine variety are precisely those that depend only upon the isomorphism class of its affine coordinate ring.*

6.6 Polynomial functions are the same as polynomial maps to the affine line

The affine coordinate ring of the affine line $\mathbb{A}^1 = k$ is a polynomial ring $k[t]$ in one variable. Theorem 26 and the map $\phi \mapsto \phi(t)$ give identifications

$$\mathrm{Hom}(X, \mathbb{A}^1) = \mathrm{Hom}_k(k[t], A) = A.$$

Thus polynomial maps $X \to \mathbb{A}^1$ are (unsurprisingly) the same as polynomial functions $X \to k$.

6.7 Continuity

Any polynomial map $f : Y \to X$ of affine varieties is continuous. In particular, polynomial functions are continuous. To see this, recall first that the topologies on X and Y are generated respectively by the basic open subsets

$$D_X(a) := \{\alpha \in X : a(\alpha) \neq 0\}, \quad D_Y(b) := \{\beta \in Y : b(\beta) \neq 0\}$$

for $a \in A := A(X), b \in B := A(Y)$. Note then that for each $a \in A$, the inverse image under f of the solution set to the inequation $a(\alpha) \neq 0$ is given by the inequation $f^\sharp(a)(\beta) = a(f(\beta)) \neq 0$, i.e.,

$$f^{-1}(D_X(a)) = D_Y(f^\sharp(a)).$$

6.8 Closed embeddings of varieties and surjective maps of algebras

In this section we generalize and further develop Examples 34, 35, 36, 37.

6.8.1 Relativized maps V_X, I_X

For an affine variety $X \subset \mathbb{A}^n = \operatorname{Specm} k[x] = \operatorname{Specm} k[x_1, \ldots, x_n]$, define the relativized-to-X variants

$$V_X : \{ \text{ subsets } S \text{ of } A(X) \ \} \to \{ \text{ closed subsets of } X \ \}$$

$$\begin{aligned} V_X(S) &:= \{\alpha \in X : a(\alpha) = 0 \text{ for all } a \in S\} \\ &= V(\pi^{-1}S), \quad \pi : k[x] \twoheadrightarrow A(X) \end{aligned}$$

and

$$I_X : \{ \text{ subsets } Y \text{ of } X \ \} \to \{ \text{ ideals of } A(X) \ \}$$

$$\begin{aligned} I_X(Y) &:= \{a \in A(X) : a(\alpha) = 0 \text{ for all } \alpha \in Y\} \\ &= \text{ image in } A \text{ of } I(Y) \end{aligned}$$

of the maps V, I defined before relative to the ambient space $\mathbb{A}^n$. As before, abbreviate $I_X(a) := I_X(\{a\})$ for $a \in A$, so that for instance $D_X(a) := \{\alpha \in X : a(\alpha) \neq 0\} = X - I_X(a)$.

Definition 39. By a *closed affine subvariety* Y of X we shall mean simply a closed subset of X regarded as an affine variety in the same ambient affine space $\mathbb{A}^n$.

Exercise 8. The maps V_X, I_X induce mutually inverse bijections

$$\{ \text{ closed affine subvarieties of } X \ \} \to \{ \text{ radical ideals of } A(X) \ \}$$

and

$$\{ \text{ irreducible closed affine subvarieties of } X \ \} \to \{ \text{ prime ideals of } A(X) \ \}$$

and

$$\{ \text{ points of } X \ \} \to \{ \text{ maximal ideals of } A(X) \ \}.$$

6.8.2 Closed embeddings

Let X be an affine variety and Y a closed affine subvariety of X. The inclusion map

$$\iota_{Y,X} : Y \hookrightarrow X$$

is a polynomial map, corresponding to the restriction map

$$\iota^\sharp_{Y,X} : A(X) \to A(Y)$$

on coordinate functions, which is surjective, inducing an identification

$$A(X)/I_X(Y) = A(Y)$$

of the coordinate ring for the closed affine subvariety Y with the quotient by the radical ideal $I_X(Y)$ of the coordinate ring for the affine variety X.

Conversely, suppose given a polynomial map $f : Z \to X$ between affine varieties with the property that the induced map $f^\sharp : A(X) \to A(Z)$ between coordinate rings is surjective. The kernel $\ker(f^\sharp) \subset A(X)$ is then the radical ideal $I_X(Y)$ corresponding to the closed affine subvariety $Y := V_X(\ker(f^\sharp))$ of X, and $f^\sharp$ factors as a composition

$$A(X) \xrightarrow{\iota^\sharp_{Y,X}} A(Y) \xrightarrow{\cong} A(Z)$$

of map between affine coordinate rings, with the second map an isomorphism, corresponding to polynomial maps

$$Z \xrightarrow{\cong} Y \xrightarrow{\iota_{Y,X}} X$$

between affine varieties, with the first map an isomorphism.

In summary, each polynomial map of affine varieties $f : Z \to X$ for which the induced map on coordinate rings $f^\sharp$ is surjective induces an isomorphism of Y with a closed affine subvariety of X. For this reason, we refer to such a map as a *closed embedding* of one affine variety into another.

6.9 Dominant maps of varieties and injective maps of algebras

Let $f : Y \to X$ be a polynomial map between affine varieties with corresponding pullback map $f^\sharp : A \to B$ on affine coordinate rings $A := A(X), B := A(Y)$.

Definition 40. The polynomial map f is called *dominant* if it has dense image.

Exercise 9. The polynomial maps in Examples 30, 31, 33 are dominant. The polynomial maps in Examples 35, 36, 38 are not dominant.

Proposition 41. *A polynomial map f is dominant if and only if $f^\sharp$ is injective.*

Proof. Note first that a basic open set $D(a)$ for $a \in A$ is nonempty if and only if $a \neq 0$. Observe then that each of the following assertions is equivalent to the next:

- f is dominant.
- For each nonempty open subset U of X, the preimage $f^{-1}(U)$ is nonempty.
- For each nonempty basic affine open subset U of X, the preimage $f^{-1}(U)$ is nonempty.
- For each nonzero $a \in A$, the preimage $f^{-1}(D(a)) = D(f^\sharp(a))$ is nonempty.
- For each nonzero $a \in A$, its image $f^\sharp(a)$ under $f^\sharp$ is nonzero.
- $f^\sharp$ is injective.

Alternatively, note first that the following conditions on a subset U of X are equivalent:

- U is dense in X, that is to say, every nonempty open subset V of X intersects U.
- For each nonzero $a \in A$, the basic open set $D_X(a)$ intersects U.
- For nonzero $a \in A$, there is some point in U at which a does not vanish.
- Any polynomial function $a \in A$ for which $a|_U$ satisfies $a = 0$.

Next, observe that a vanishes on image(f) if and only if $f^\sharp(a) = a \circ f = 0$. Thus the following are equivalent:

- f is dominant, that is to say, image(f) is dense in X.
- Any polynomial function $a \in A$ for which $a|_{\text{image}(f)} = 0$ satisfies $a = 0$.
- Any polynomial function $a \in A$ for which $f^\sharp(a) = 0$ satisfies $a = 0$, i.e., $f^\sharp$ is injective.

□

Example 42. Let us use the above reasoning to give a mildly different proof that an affine variety X is irreducible if and only if $I(X)$ is a prime ideal, or equivalently, if and only if its affine coordinate ring $A := A(X)$ is an integral domain. Indeed, each of the following is equivalent to the next:

- X is irreducible.
- Every nonempty open subset of X is dense.
- Every nonempty basic open subset $D(a)$ of X $(a \in A - \{0\})$ is dense.
- Every basic affine open inclusion $\iota_{X,a} : X_a \to X$ with nonempty image $(a \in A - \{0\})$ is dominant.
- Every localization $\iota_{X,a}^\sharp : A \to A_a$ with $a \in A - \{0\}$ is injective.

- Every $a \in A - \{0\}$ is a non-zerodivisor.
- A is an integral domain.

(See also Example 38.)

Remark 43. "Dominant closed embeddings are isomorphisms" is the (rather intuitively plausible) geometric translation of the assertion that an injective and surjective morphism of rings (or of finite type reduced k-algebras) is an isomorphism.

6.10 Localization as the inclusion of the complement of a hypersurface

6.10.1 Definition and informal discussion

Here we generalize Example 33 with the aim of giving a geometric interpretation of localization maps of rings. Let $X \subset \mathbb{A}^n := \operatorname{Specm} k[x], k[x] := k[x_1, \ldots, x_n]$ be an affine variety in n-dimensional affine space with affine coordinate ring $A := A(X) = k[x]/I(X)$. A good first example to consider is when $X = \mathbb{A}^n$ is the entire space, even when $n = 1$.

Definition 44. To each polynomial function $a \in A$ on X we attach the affine variety

$$X_a \subset \mathbb{A}^{n+1} := \operatorname{Specm} k[x, z], \quad k[x, z] := k[x_1, \ldots, x_n, z]$$

in $(n + 1)$-dimensional affine space cut out by the defining equations for X in the first n variables together with the equation $az = 1$, thus

$$X_a := \{(\alpha, \beta) := (\alpha_1, \ldots, \alpha_n, \beta) \in k^{n+1} : \alpha \in X, a(\alpha)\beta = 1\},$$

as well as the map

$$\iota_{X,a} : X_a \to X$$
$$(\alpha, \beta) \mapsto (\alpha)$$

induced by projection onto the first n coordinates.

The reader is encouraged now to review Example 33 to see how it fits into this more general picture.

The maps $\iota_{X,a}$ are the most important ones between affine varieties that we shall consider, so we accord them a detailed discussion below in the proof of Proposition 46. Before that, some informal remarks intended to help orient the reader:

- X_a is an affine variety, as it is defined by polynomial equations.
- Since $\iota_{X,a}$ is given by the polynomial function $(\alpha_1, \ldots, \alpha_n, \beta) \mapsto (\alpha_1, \ldots, \alpha_n)$ in the coordinate variables, it is a polynomial map.

- For any pair $(\alpha, \beta) \in X^a$ we have $a(\alpha) \neq 0$, and the second coordinate β is determined by the first via $\beta = 1/a(\alpha)$, so it is clear that $\iota_{X,a}$ induces a bijection
$$X_a \xrightarrow{\cong} D_X(a) := \{\alpha \in X : a(\alpha) \neq 0\}$$
onto the complement in X of the locus of a. These sets may be identified without too much harm.

- The ring $A(X_a)$ of polynomial functions on X_a is generated by the ring of polynomial functions on X together with the new function
$$z : X_a \ni (\alpha, \beta) \mapsto \beta = 1/a(\alpha) \in k$$
obtained by taking the $(n+1)$st coordinate. It follows that
$$A(X_a) = A[1/a] = A_a$$
"is" the localization of A at a, and the pullback map $\iota_{X,a}^{\sharp}$ "is" the localization map $A \to A_a$.

One should have in mind something like the following table:

geometry	algebra
affine variety $X \subset \mathbb{A}^n$	affine coordinate ring $A \twoheadleftarrow k[x_1, \ldots, x_n]$
discarding from X the locus of some $a \in A$	replacing A with its localization A_a
restricting a polynomial function on X to $D_X(a)$	taking its image under $A \to A_a$
extending it from $D_X(a)$ back to X	finding a preimage under $A \to A_a$
a polynomial function on X vanishes on $D_X(a)$	it belongs to $\ker(A \to A_a) = \cap_{N \geq 0} \operatorname{ann}(a^N)$
realizing $D_X(a)$ as the affine variety $X_a \subset \mathbb{A}^{n+1}$	writing $A_a = A[z]/(1 - az)$
the function $\alpha \mapsto f(\alpha)/a(\alpha)^N$ on $D_X(a)$	the element f/a^N of $A_a \cong A(X_a)$
the polynomial $(\alpha, \beta) \mapsto f(\alpha)\beta^N$ on X_a	the element f/a^N of $A_a \cong A(X_a)$

Definition 45. The polynomial maps $\iota_{X,a}$ shall be referred to as *basic affine open inclusions* into X and their images $D_X(a)$ as *basic affine open subsets* of X, or simply as *basic affine opens* of X; the latter form an open basis for the Zariski topology and are homeomorphic to the affine varieties X_a via the maps $\iota_{X,a}^{-1}$, justifying the terminology. We write $D(a)$ instead of $D_X(a)$ for $a \in A := A(X)$ when the ambient variety X is clear from context.

6.10.2 Formal discussion

For the sake of clarity and completeness, and because of the vital importance of understanding this class of maps properly, we record here a stiff, formal development of the ideas explained informally in the previous section. The reader should not be tricked into thinking that following result is difficult or deep.

Proposition 46. *X_a is an affine variety. $\iota_{X,a}$ is a polynomial map. $\iota_{X,a}$ is injective, and induces a homeomorphism onto its image*

$$\operatorname{image}(\iota_{X,a}) = D_X(a).$$

Denote by

$$A_a := \frac{A[z]}{(1 - az)}$$

the localization of A at a, regarded as a quotient of $k[x,z]$. Then the natural restriction map

$$k[x,z] \twoheadrightarrow A(X_a)$$

factors through a well-defined isomorphism

$$A_a \xrightarrow{\cong} A(X_a) \tag{8}$$

forming a commutative triangle with the localization map $A \to A_a$ and the pullback map $\iota_{X,a}^\sharp : A \to A(X_a)$.

Proof. The first three assertions are proved exactly as in the previous section. By definition, X_a is the affine variety $V(\mathfrak{b})$, where $\mathfrak{b}$ is the ideal

$$\mathfrak{b} := (I(X), 1 - \tilde{a}z) \subset k[x,z]$$

with $\tilde{a} \in k[x]$ any representative for $a \in A$. The map $k[x,z] \twoheadrightarrow A_a$ factors as the composition $k[x,z] \twoheadrightarrow A[z] \twoheadrightarrow A_a$, and we have

$$\ker(k[x,z] \twoheadrightarrow A[z]) = (I(X)) \subset k[x,z],$$

$$\ker(A[z] \twoheadrightarrow A_a) = (1 - az) \subset A[z],$$

hence $\ker(k[x,z] \twoheadrightarrow A_a) = \mathfrak{b}$. Since $\mathfrak{b} \subset I(V(\mathfrak{b})) = I(X_a) = \ker(k[x,z] \twoheadrightarrow A(X_a))$, the map (8) is well-defined, and is an isomorphism if and only if $\mathfrak{b} = I(V(\mathfrak{b}))$, or equivalently (by the consequence $I(V(\mathfrak{b})) = r(\mathfrak{b})$ of the Nullstellensatz) if and only if the quotient ring $A_a \cong k[x,z]/\mathfrak{b}$ is reduced. That this is so follows from the more general fact (left as an exercise to the reader) that any localization of a reduced ring is reduced. The remaining assertion to be verified is that $\iota_{X,a}$ induces a homeomorphism onto its image. Continuity follows from the fact that $\iota_{X,a}$ is a polynomial map. Conversely, an open base on X_a is given by the sets $D_{X_a}(f) := \{\gamma \in X_a : f(\gamma) \neq 0\}$ with $f \in A(X_a) \cong A_a$. Write $f = g/a^N$ for some $g \in A$ and $N \in \mathbf{Z}_{\geq 0}$. Then each of the following conditions on $\alpha \in X$ is equivalent to the next:

- $\alpha \in \iota_{X,a}(D_{X_a}(f))$.
- $a(\alpha) \neq 0$ and $g(\alpha)/a(\alpha)^N \neq 0$.
- $g(\alpha)a(\alpha) \neq 0$.
- $\alpha \in D_X(ga)$.

Therefore $\iota_{X,a}$ maps basic open sets to basic open sets, and so defines a homeomorphism onto its image. □

6.10.3 Some very mild cautions concerning "identifications"

We shall often identify $A(X_a) = A_a$ via the isomorphism described above, so that the pullback map $\iota_{X,a}^\sharp : A \to A(X_a)$ "is" the localization map

$$\iota_{X,a}^\sharp : A \to A_a.$$

We might also (rather harmlessly) confuse X_a with its image $D_X(a)$, but caution that:

1. The open subset $D_X(a)$ of the affine variety X only itself attains the structure of an affine variety in the strict sense in which we have defined it (i.e., coming with some chosen closed embedding into affine space) after adding the extra variable z and realizing it in the one-dimension-up affine space.

2. It is in many respects healthiest to view the inclusions $\iota_{X,a} : X_a \to X$ as more fundamental than their images $D_X(a)$.

6.10.4 Extreme cases

Suppose in the above that a is a constant function, say $a = 1$. Then its locus is the empty set, whose complement is thus the entire space $D_X(a) = X$. The space X_a is just $\{(\alpha, 1) : \alpha \in X\}$ and the map $\iota_{X,a} : X_a \to X$ is an isomorphism corresponding on coordinate rings to the isomorphism $\iota_{X,a}^\sharp : A \to A_a$. (Similar assertions apply more generally when a is a unit.)

At the other extreme, if $a = 0$, then $D_X(a) = X_a = \emptyset$, corresponding to the fact that the localization A_a is the zero ring. (Similar assertions apply more generally when a is nilpotent, and to a lesser extent when a is a zerodivisor.)

6.10.5 Compatibilities when one repeatedly localizes

It is often the case that one is interested in throwing away the locus of not just one polynomial function, but also that of (erm, well...) some *other* polynomial function. For instance, starting with the affine plane $X := \mathbb{A}^2 := \operatorname{Specm} k[x_1, x_2]$, one could throw away the horizontal "x_1-axis" $V_X(x_2)$ and then throw away the vertical "x_2-axis" $V_X(x_1)$, or perform the same operations in the opposite order, but in either case, one is left with the complement $D_X(x_1 x_2) = D_X(x_1) \cap D_X(x_2)$ of the cross obtained as union of the horizontal and vertical axes.

More generally, consider an affine variety $X \subset \mathbb{A}^n$ with affine coordinate ring $A := A(X)$, and let $a_1, a_2 \in A$ be two polynomial functions. Then

$$D_X(a_1 a_2) = D_X(a_1) \cap D_X(a_2),$$

that is to say, if we throw away both the set where a_1 vanishes and the set where a_2 vanishes, what we're left with is the set where $a_1 a_2$ does not vanish.

We now have two routes for including the subset $D_X(a_1a_2)$ back into X, either first through $D_X(a_1)$ as

$$D_X(a_1a_2) \subset D_X(a_1) \subset X$$

or first through $D_X(a_2)$ as

$$D_X(a_1a_2) \subset D_X(a_2) \subset X,$$

which correspond upon identifying each basic open $D_X(a)$ with the corresponding affine variety X_a for two ways to map $X_{a_1a_2} \to X$, namely as

$$X_{a_1a_2} \to X_{a_1} \to X$$

$$(\alpha, \frac{1}{a_1a_2(\alpha)}) \mapsto (\alpha, \frac{1}{a_1(\alpha)}) \mapsto (\alpha)$$

$$(\alpha, \beta) \mapsto (\alpha, a_2(\alpha)\beta) \mapsto (\alpha)$$

or as

$$X_{a_1a_2} \to X_{a_2} \to X$$

$$(\alpha, \frac{1}{a_1a_2(\alpha)}) \mapsto (\alpha, \frac{1}{a_2(\alpha)}) \mapsto (\alpha)$$

$$(\alpha, \beta) \mapsto (\alpha, a_1(\alpha)\beta) \mapsto (\alpha).$$

Each of the above arrows is visibly a polynomial map, being given in at least one of its representations by an expression that is visibly polynomial in the coordinate functions. Moreover, both of the compositions $X_{a_1a_2} \to X$ visibly coincide with each other and also with the map $\iota_{X,a_1a_2} : X_{a_1a_2} \to X$. In terms of coordinate rings, these polynomial maps correspond to the localization maps

$$A \to A_{a_1} \to A_{a_1a_2}$$

$$A \to A_{a_2} \to A_{a_1a_2},$$

each of which coincide with the localization map $A \to A_{a_1a_2}$. The geometric interpretation of the (trivial) coincidence of these localization maps is the thus (the triviality) that for a function f on X, the following functions on $D_X(a_1a_2)$ are the same:

- The restriction of f to $D_X(a_1a_2)$.
- The restriction to $D_X(a_1a_2)$ of the restriction of f to $D_X(a_1)$.
- The restriction to $D_X(a_1a_2)$ of the restriction of f to $D_X(a_2)$.

We record below some notation to accompany these and other trivial coincidences, which we shall occasionally use without explicit mention.

Exercise 10. For a function f with domain a subset of X that contains $D_X(a)$, define $f|_{X_a}$ to be its pullback under $\iota_{X,a}$, i.e.,

$$f|_{X,a} := f \circ \iota_{X,a},$$

which we shall refer to loosely as the *restriction* of f to X_a. Thus, for instance,

$$f \in A \implies f|_{X_a} = \iota_{X,a}^{\sharp}(f) \in A_a.$$

Similarly, for f with domain a subset of X_{a_1} containing $D_{X_{a_1}}(a_2)$, define $f|_{X_{a_1 a_2}}$ by pulling back under the map $X_{a_1 a_2} \to X_{a_1}$ described above, that is,

$$f|_{X_{a_1 a_2}} := f \circ (X_{a_1 a_2} \to X_{a_1}).$$

Verify that the following identities hold whenever they make sense:

$$(f|_{X_{a_1}})|_{X_{a_1}} = f|_{X_{a_1}},$$

$$(f|_{X_{a_1}})|_{X_{a_2}} = (f|_{X_{a_2}})|_{X_{a_1}} = f|_{X_{a_1 a_2}},$$

$$f|_{X_{a_1 a_2 a_3}} = (f|_{X_{a_1 a_2}})|_{X_{a_1 a_2 a_3}} = (f|_{X_{a_1 a_3}})|_{X_{a_1 a_2 a_3}}.$$

6.10.6 Functoriality

Recall that localization is a functor, i.e., that for each ring morphism $\phi : A \to B$ and multiplicative subset S of A one obtains a natural ring morphism $S^{-1}\phi : S^{-1}A \to S^{-1}B$ of localized rings. The geometric incarnation of this functoriality is given by restricting a polynomial map between affine varieties to a preimage of a basic open subset of the codomain:

Exercise 11. Let $f : Y \to X$ be a polynomial map of affine varieties. Denote by $A := A(X), B := A(Y)$ their coordinate rings. Let $a \in A$.

1. Verify that $f^{-1}(D_X(a)) = D_Y(f^{\sharp}(a))$.
2. Verify that there exists a unique polynomial map $f_a : Y_{f^{\sharp}(a)} \to X_a$ forming a commutative square

$$\begin{array}{ccc} Y_{f^{\sharp}(a)} & \xrightarrow{f_a} & X_a \\ {\scriptstyle \iota_{Y,f^{\sharp}(a)}}\downarrow & & \downarrow{\scriptstyle \iota_{X,a}} \\ Y & \xrightarrow[f]{} & X. \end{array}$$

3. Verify that $f_a^{\sharp} : A_a \to B_{f^{\sharp}(a)}$ is the localization of $f^{\sharp}$ at a (i.e., $f_a^{\sharp}$ is the natural map induced by $f : A \to B$ and $1/a \mapsto 1/f^{\sharp}(a)$).

Example 47. Take

$$X := \mathbb{A}^1 := \operatorname{Specm} k[x_1],$$

$$Y := V(y_1 y_2) \subset \mathbb{A}^2 := \operatorname{Specm} k[y_1 y_2],$$

and

$$Y \xrightarrow{f} X$$
$$(\beta_1, \beta_2) \mapsto (\beta_1).$$

Thus f is the projection onto the horizontal axis from the cross obtained as the union of the vertical and horizontal axes. For the sake of illustration, let us record that the fibers of this map are

$$f^{-1}(\alpha_1) = \begin{cases} \{(\alpha_1, 0)\} & \text{if } \alpha_1 \neq 0 \\ \{(0, \gamma) : \gamma \in k\} & \text{if } \alpha_1 = 0. \end{cases}$$

The pullback map on coordinate rings is

$$A = k[x_1] \xrightarrow{f^\sharp} B = \frac{k[y_1, y_2]}{(y_1 y_2)}$$
$$x_1 \mapsto y_1.$$

Let us localize at the element $a := x_1 \in A$, so that $f^\sharp(a) = y_1$,

$$D_X(a) = \{(\alpha_1) : \alpha_1 \in k - \{0\}\}$$
$$D_Y(f^\sharp(a)) = \{(\beta_1, 0) : \beta_1 \in k - \{0\}\}$$
$$X_a = \{(\alpha_1, \gamma) : \alpha_1 \gamma = 1\}$$
$$Y_{f^\sharp(a)} = \{(\beta_1, 0, \gamma) : \beta_1 \gamma = 1\}$$

and

$$Y_{f^\sharp(a)} \xrightarrow{f_a^\sharp} X_a$$
$$(\beta_1, 0, \gamma) \mapsto (\alpha_1, \gamma)$$

or

$$D_Y(f^\sharp(a)) \to D_X(a)$$
$$(\beta_1, 0) \mapsto (\alpha_1)$$

corresponding to

$$A_a = \frac{k[x_1, z]}{(1 - x_1 z)} \cong k[x_1, 1/x_1] \xrightarrow{f_a^\sharp} B_{f^\sharp(a)} = \frac{k[y_1, y_2, z]}{(y_1 y_2, 1 - y_1 z)} \cong \frac{k[y_1, y_2, 1/y_1]}{(y_1 y_2)}$$
$$x_1 \mapsto y_1$$
$$z \mapsto z.$$

Since y_1 is a unit in $k[y_1, y_2, 1/y_1]$, we have

$$\frac{k[y_1, y_2, 1/y_1]}{(y_1 y_2)} \cong \frac{k[y_1, y_2, 1/y_1]}{(y_2)} \cong k[y_1, 1/y_1],$$

hence $f_a^\sharp$ is an isomorphism. Geometrically, this makes precise the assertion that f is an "isomorphism away from the vertical axis."

Exercise 12. Let

$$Y := \mathbb{A}^1 := \operatorname{Specm} k[y_1],$$

$$X := V(x_1 x_2) \subset \mathbb{A}^2 := \operatorname{Specm} k[x_1, x_2],$$

$$Y \xrightarrow{f} X$$

$$(\beta_1) \mapsto (\beta_1, 0)$$

be the inclusion of the horizontal axis into the union of the horizontal and vertical axes. Carry out calculations as in Example 47 concerning the localized polynomial maps f_a when

1. $a := (x_2)$ and
2. $a := (x_1)$.

Draw pictures.

6.11 Notable omission: the localization map at a prime ideal

Among the most fundamental maps out of a ring A are

- the localization maps $A \to A_a = A[1/a]$ at elements $a \in A$,
- the quotient maps $A \twoheadrightarrow A/\mathfrak{a}$ by ideals $\mathfrak{a}$, and
- the localization maps $A \to A_\mathfrak{p}$ at prime ideals $\mathfrak{p}$.

When A is a finite type reduced k-algebra (arising as the affine coordinate ring of some affine k-variety X), we have seen in the preceeding sections some geometric interpretations of the first two sorts of maps, with the localizations $A \to A_a$ corresponding to basic affine open inclusions $X_a \to X$ onto the complements of hypersurfaces $V_X(a)$ and the quotients $A \to A/\mathfrak{a}$ corresponding to closed embeddings $Z \hookrightarrow X$ with image the closed subvariety $V_X(\mathfrak{a})$ cut out by $\mathfrak{a}$, but not of the third sort. There are good reasons:

- Localizations $A_\mathfrak{p}$ at primes are typically not finite type k-algebras, and so do not arise as affine coordinate rings of affine varieties. For example, the localization of $k[x]$ at (x) is not finite type.
- Being local rings, their maximal spectra $\operatorname{Specm}(A_\mathfrak{p}) = \{\mathfrak{p}_\mathfrak{p}\}$ are singletons consisting of the extension $\mathfrak{p}_\mathfrak{p}$ of the prime ideal $\mathfrak{p}$.

Nevertheless, the ring morphisms $A \to A_\mathfrak{p}$ and the *prime* spectra of the rings $A_\mathfrak{p}$ are of geometric significance, e.g.:

Exercise 13. For X an affine variety with affine coordinate ring A and $Z \subset X$ an irreducible closed affine subvariety (e.g., a point) with vanishing (prime) ideal $\mathfrak{p} := I_X(Z) \subset A$, show that the natural maps

$$\operatorname{Spec}(A_{\mathfrak{p}}) \to \{ \text{ irreducible closed subvarieties of } X \text{ containing } Z \ \}$$

$$\mathfrak{p}_1 \mapsto V_X(\text{contraction of } \mathfrak{p}_1 \text{ to } A)$$

$$\text{extension of } I_X(Y) \text{ to } A_{\mathfrak{p}} \ \leftarrow\!\shortmid Y$$

are mutually inverse inclusion-reversing bijections.

7 Partitions of unity

7.1 Statement of result

Definition 48. By a *basic cover* of an open subset U of an affine variety X we shall mean the datum of a family of elements $(a_i)_{i \in I} \subset A(X)$ so that $U = \cup_{i \in I} D_X(a_i)$. We denote this datum symbolically by

$$(X_{a_i} \to U \subset X)_{i \in I}$$

or in the special case $U = X$ simply by

$$(X_{a_i} \to X)_{i \in I}.$$

Example 49. Let $X := \mathbb{A}^2 := \operatorname{Specm} k[x_1, x_2]$ and $U := \mathbb{A}^2 - \{(0,0)\}$. Set $I := \{1, 2\}$ and $a_1 := x_1, a_2 := x_2$. Then $(X_{a_i} \to U \subset X)_{i \in I}$ is a basic cover of U.

The aim of this section is to establish the following fundamental result about polynomial functions on affine varieties:[8]

Theorem 50. *Let X be an affine variety, denote by $A := A(X)$ its affine coordinate ring, and let $(X_{a_i} \to X)_{i \in I}$ be a basic cover of X. For each family $(s_i)_{i \in I}$ of polynomial functions $s_i \in A(X_{a_i}) = A_{a_i}$ on the X_{a_i} satisfying the overlap condition*

$$s_i|_{X_{a_i a_j}} = s_j|_{X_{a_i a_j}},$$

there exists a unique $s \in A$ so that $s|_{X_{a_i}} = s_i$ for all i.

The case $X = \mathbb{A}^n$ is already interesting and worth considering first. The proof is a mildly elaborate exercise in localization which we shall carry out slowly and deliberately in order to emphasize its geometric content.

[8] The core statement underlying this result has nothing to do with finite type k-algebras or varieties; it is a basic fact about rings A and their localizations A_{a_i} (see Exercise 3.24 in Atiyah–Macdonald or Hartshorne, II, Proposition 2.2). Our proof should make this point clear, although we will not belabor it.

7.2 The Zariski topology on an affine variety is noetherian

Let X be an affine variety with affine coordinate ring A, a finite type reduced k-algebra. By the Hilbert basis theorem, A is a noetherian ring. This means that every ascending chain of ideals stabilizes, or equivalently that every ideal of A is finitely-generated. We record here the routine translation of these algebraic properties of A into geometric properties of X, where as usual each of the following assertions should be evidently equivalent to the next:

- Every ascending chain $\mathfrak{a}_1 \subset \mathfrak{a}_2 \subset \cdots\cdots$ of radical ideals in A stabilizes.
- Every descending chain $Y_1 \supset Y_2 \supset \cdots$ of closed subsets of X stabilizes.
- Every ascending chain $U_1 \subset U_2 \subset \cdots$ of open subsets of X stabilizes.

Now we do the same thing, using the criterion involving finite generation of ideals:

- Every radical ideal $\mathfrak{a}$ of A is finitely generated.
- For each radical ideal $\mathfrak{a}$ of A there exists elements $a_1, \dots, a_k \in A$ so that $\mathfrak{a} = (a_1, \dots, a_k)$.
- For each closed subset Y of X there exist $a_1, \dots, a_k \in A$ so that $Y = V_X(\{a_1, \dots, a_k\})$, the latter of which we pause to recall is equal to the intersection $V_X(a_1) \cap \cdots \cap V_X(a_k)$. In words, every subvariety is defined by the conjunction of finitely many equations.
- For each open subset U of X there exist $a_1, \dots, a_k \in A$ so that $U = D_X(a_1) \cup \cdots \cup D_X(a_k)$. In words, every open set is defined by the disjunction of finitely many inequations.

In particular, *any open subset of an affine variety is a finite union of basic affine opens.*

Definition 51. A topological space X is called

- *noetherian* if every ascending chain of open subsets stabilizes, and
- *quasicompact* if every open cover admits a finite subcover.[9]

With this terminology, and bearing in mind that (by definition) the sets $D_X(a)$ for $a \in A$ generate the topology on X, the translation of the Hilbert basis theorem carried out above asserts that *every affine variety X is noetherian and every open subset U of X is quasicompact.*

Exercise 14. Show that a topological space is noetherian if and only if every open subset is quasicompact. (This exercise provides the topological/geometric translation of the equivalence between the two characterizations of noetherian rings recalled above.)

[9] *Compact* topological spaces are traditionally required to be Hausdorff.

7.3 Basic covers of an affine variety yield partitions of unity

By the discussion of Section 7.2, each basic cover $(X_{a_i} \to X)_{i \in I}$ of an affine variety X remains such upon upon restricting to some *finite* subset of I. This fact can be seen more explicitly and more usefully through the following evident chain of equivalences:

- $(X_{a_i} \to X)_i$ is a basic cover.
- $X = \cup_i D_X(a_i)$.
- $\emptyset = \cap_i V_X(a_i)$.
- $\emptyset = V_X(\mathfrak{a})$, where $\mathfrak{a} := (\{a_i\}_i)$ is the ideal generated by the a_i.
- $\mathfrak{a} = (1)$. (Here we have used the Nullstellensatz.)
- There exists
  - a finite subset of I, which we label as $\{1, 2, \dots, k\}$ for notational simplicity, and
  - coefficients $c_1, \dots, c_k \in A$

  so that
  $$1 = c_1 a_1 + \cdots + c_k a_k \tag{9}$$

In other words, every basic cover comes with a *partition of unity* as in (9), which gives a particularly strong certificate of the covering property. The terminology is justifed by noting that the term $c_i a_i$ vanishes on $V_X(a_i)$ and hence is supported on the open subset $D_X(a_i)$ of the open cover $X = D_X(a_1) \cup \cdots \cup D_X(a_k)$. By multiplication, it follows that any function $f : X \to k$ may be written as a sum

$$f = f_1 + \cdots f_k$$

where $f_i := f c_i a_i$ is supported set-theoretically on $D_X(a_i)$.

Let us note the following final equivalent of the above sequence of characterizations of a basic cover $(X_{a_i} \to X)_i$:

- There exists a finite subset $\{1, 2, \dots, k\} \subset I$ so that for each positive integer N there exist coefficients $c_1, \dots, c_k \in A$ so that
  $$1 = c_1 a_1^N + \cdots + c_k a_k^N. \tag{10}$$

The equivalence holds because $(a_1, \dots, a_k) = 1$ if and only if $(a_1^N, \dots, a_k^N) = 1$, or alternatively by raising the identity (10) to a sufficiently large power (and then renaming the coefficients $c_i \in A$ suitably). As we shall shortly see, the partitions (10) can be useful with N taken large because then (informally speaking) the a_i^N vanish to a greater extent on $V_X(a_i)$, which equips them better to kill the denominators of what they are multiplied against.

7.4 Proof of Theorem 50

Exercise 15. Do or attempt Exercise 3.24 in Atiyah–Macdonald, if you haven't.[10]

The uniqueness assertion in Theorem 50 follows from the fact that the $D_X(a_i)$ cover X.[11]

In proving the existence part of Theorem 50, we may reduce to the case I is finite, or only consider that case in the first place, as it suffices for all applications.[12] Suppose thus that $I = \{1, \dots, k\}$ is finite. The basic idea is to use a partition of unity (10) with N sufficiently large to extend suitable multiples of the local polynomial functions s_i on the X_{a_i} to global polynomial functions on X whose sum s satisfies $s|_{X_{a_i}} = s_i$. The largeness of N is used both in the extension and the verification.

To implement this idea rigorously, we begin by writing down what we know about the s_i. We identify them with elements of A_{a_i} and write them as

$$s_i = \frac{h_i}{a_i^N}$$

for some $h_i \in A$ and some $N \in \mathbf{Z}_{\geq 0}$ which we may be chosen sufficiently large to work simultaneously for all i in the finite index set I. By the overlap condition $s_i|_{X_{a_i a_j}} = s_j|_{X_{a_i a_j}}$, there exists $M \in \mathbf{Z}_{\geq 0}$ (taken large enough to work for all pairs i, j) so that

$$(a_i a_j)^M (h_i a_j^N - h_j a_i^N) = 0.$$

By replacing h_i with $a_i^M h_i$ and N with $N + M$, we may and shall reduce to the case $M = 0$, in which we are given that

$$s_i = \frac{h_i}{a_i^N} \text{ with } h_i \in A \text{ and } h_i a_j^N = h_j a_i^N.$$

These two equations imply that for a partition of unity

$$1 = \sum c_i a_i^N$$

with $c_i \in A$, the desiderata that the identities

$$s|_{X_{a_j}} = s_j \text{ for each } j \tag{11}$$

are satisfied upon taking for s the sum

$$s := \sum f_i$$

[10] Consider also glancing at the proof of Proposition 2.2 in Hartshorne.

[11] Alternatively, one can argue using an algebraic partition of unity as above or as below.

[12] Let $I_0 \subset I$ be a finite subset for which $(X_{a_i} \to X)_{i \in I_0}$ remains a basic cover, and assume the theorem holds for any larger finite index set $I_1 \supset I_0$. By the uniqueness, the polynomial function $s \in A$ produced by the theorem is independent of I_1, and so is seen to satisfy $s|_{X_{a_i}} = s_i$ for each $i \in I$ upon taking (for instance) $I_1 := I_0 \cup \{i\}$.

of the polynomial functions

$$f_i := h_i c_i \in A,$$

whose definition we motivate by the calculation

$$c_i a_i^N s_i = c_i a_i^N \frac{h_i}{a_i^N} = h_i c_i$$

in the localized ring A_a. To establish (11), it suffices by the nonvanishing of a_j on X_{a_j} (or alternatively, by properties of localization) to verify that

$$a_j^N s|_{X_{a_j}} = a_j^N s_j,$$

which we obtain as follows using the overlap condition and the definitions of s and h_j:

$$a_j^N s|_{X_{a_j}} = a_j^N \sum_i h_i c_i = h_j \sum_i a_i^N c_i = h_j = a_j^N s_j.$$

This completes the proof of the theorem.

8 Regular functions on open subsets of an affine variety

8.1 Is "being a polynomial function" a local notion?

The principal aim of this section is to address the question of the extent to which the definitions given hitherto (of polynomial functions and polynomial maps) have been *local*, that is to say, the extent to which they depend only upon behavior in small open neighborhoods (where "local" and "open" always mean with respect to the Zariski topology). Take for instance an affine variety $X \subset \mathbb{A}^n$ with coordinate ring $A := A(X)$ and an open subset $U \subset X$ – the case $U = X = \mathbb{A}^n$ will already be interesting – and consider the following conditions on a function $f : U \to k$:

(R0) **Being a polynomial function.**
(This condition should be considered only when $U = X$.) f is a polynomial function.

(R1) **Basic affine condition.**
For each $a \in A$ for which the basic open subset $D_X(a)$ is contained in U, the restriction

$$f|_{X_a} := f \circ \iota_{X,a} : X_a \to k$$

of f to X_a defines a polynomial function, that is to say, $f|_{X_a}$ belongs to the localization A_a subject to our standard identifications

$$A_a = A(X_a) \subset \mathrm{Func}(X_a, k).$$

(R2) **Zariski local condition.**
For each point $\alpha \in U$ there exists an open neighborhood $\alpha \in V \subset U$ and polynomial functions $g, h \in A$ for which $h(v) \neq 0$ for all $v \in V$ and so that on V, the identity $f = g/h$ holds.

(R3) **Basic affine local condition.**
There exists a basic cover (see Definition 48)

$$(X_{a_i} \to U \subset X)_{i \in I},$$

of U with the property that $f|_{X_{a_i}} \in A_{a_i}$ for each index $i \in I$.

Example 52. When $U = X = \mathbb{A}^n$:

- Condition (R0) says that f is given *globally*, that is to say, everywhere simultaneously, by a polynomial in the coordinate variables.
- Condition (R1) says that f is given on each basic open set $D(a) = \{\alpha \in X : a(\alpha) \neq 0\}$ by some ratio $f = g/a^N$ with $g \in A$ and $N > 0$. Since we allow the case $a = 1$, this is equivalent to condition (R0).
- Condition (R2) says that f is *locally*, i.e., on an open neighborhood of each point, a ratio of polynomials, with the denominator nonvanishing on that neighborhood so that the ratio makes sense.
- Condition (R3) says that the pullback of f to a basic cover is polynomial. (This assertion is effectively a refinement of condition (R2) in which we restrict to the basic open sets and formulate things in terms of covers rather than points.)

Theorem 53. *The conditions (R0), (R1), (R2), (R3) are equivalent.*

Note that if one replaces "polynomial function" in the above conditions with "continuous function," then their equivalence becomes completely obvious.

Proof. The only nontrivial implication is that (R3) implies (R1), but let us carefully verify them all:

- (R0) implies (R1) when $U = X$: If $f \in A$ is polynomial function on X, then since $\iota_{X,a}$ is a polynomial map, we have $f|_{X_a} = \iota_{X,a}^{\sharp}(f) \in A_a$.
- (R1) implies (R0) when $U = X$: Take $a = 1$, so that $D_X(a) = X$ and $\iota_{X,a} : X_a \to X$ is an isomorphism.
- (R1) implies (R2): Let $\alpha \in U$. Choose $a \in A$ so that $\alpha \in D_X(a) \subset U \subset X$. By (R1), $f|_{X^a} = g/a^N \in A_a$ for some $g \in A, N \in \mathbf{Z}_{\geq 0}$. Take $U := D_X(a), h := a^N$. Then (R2) is satisfied.

- (R2) implies (R3): For each $\alpha \in U$, choose a neighborhood $\alpha \in V \subset U$ and $g, h \in A$ such that $h(v) \neq 0$ for $v \in V$ and $f = g/h$ holds on V. Choose $k \in A$ for which $\alpha \in D_X(k) \subset V$. Set $a_\alpha := hk \in A$. Then

  $$\alpha \in D_X(a_\alpha) = D_X(h) \cap D_X(k) \subset V.$$

  By the identity $f = g/h$ on V,

  $$f|_{X_{a_\alpha}} = gk/a_\alpha \in A_{a_\alpha}.$$

  Therefore (R3) holds with the basic cover $(X_{a_\alpha} \to U \subset X)_{\alpha \in U}$.

- (R3) implies (R1): Let $(X_{a_i} \to U \subset X)_i$ be a basic cover as in (R3). Let $D_X(b) \subset U$ ($b \in A$) be a basic open subset as in (R1). We must show that $f|_{X_b}$ is polynomial. The natural maps $X_{a_i b} \to X_b$ as in Section 6.10.5 furnish a basic cover $(X_{a_i b} \to X_b)_i$ of X_b. The functions $s_i := f|_{X_{a_i b}} = (f|_{X_{a_i}})|_{X_{a_i b}}$ are polynomial by the assumption that $f|_{X_{a_i}}$ is polynomial, and they satisfy the overlap condition, since

  $$s_i|_{X_{a_i a_j b}} = f|_{X_{a_i a_j b}} = s_j|_{X_{a_i a_j b}}.$$

  Theorem 50 produces a (unique) polynomial function s on X_b so that $s|_{X_{a_i b}} = s_i$ and hence

  $$(f|_{X_b} - s)|_{X_{a_i b}} = 0$$

  for each i. Since the $X_{a_i b}$ cover X_b, it follows that $f|_{X_b} - s = 0$, whence $f|_{X_b}$ is polynomial, as required.

□

8.2 Definition and basic properties of regular functions

Definition 54. Let X be an affine variety. For each open subset U of X, denote by $\mathcal{O}(U)$ the space of functions $f : U \to k$ satisfying any of the equivalent conditions (R1) through (R3) from Section 8.1. Elements of $\mathcal{O}(U)$ shall be called *regular functions* on U.

Remark 55.

1. Let X be an affine variety, Y a closed affine subvariety of X, and U an open subset both of Y and of X. The restrictions of polynomial functions on Y to U are the same as restrictions of polynomial functions on X to U. Since a regular function on U is (by (R2)) just a function which is locally a quotient of restrictions of polynomial functions, it follows that the notion of a regular function on U is independent of whether U is regarded as an open subset of Y or of X.

2. By the equivalence of (R1) with (R0) when $U = X$, we have

   $$\mathcal{O}(X) = A(X).$$

In words, this is the nontrivial assertion established in Section 7 that *every regular function on an affine variety is a polynomial function.* (The importance of this assertion is the reason for which we postponed the definition of "regular function" until after we proved Theorem 53.)

3. By (R1), we also have for each $a \in A(X)$ that
$$\mathcal{O}(D_X(a)) \cong A(X)_a.$$
In words, the *regular functions on a basic affine open subset $D_X(a)$ of X correspond under the natural bijection $\iota_{X,a} : X_a \to D_X(a)$ to the polynomial functions on the corresponding affine variety X_a,* with the regular function
$$f/a^N : D_X(a) \ni \alpha \mapsto f(\alpha)/a(\alpha)^N$$
corresponding to the polynomial function
$$f/a^N : X_a \ni (\alpha, \beta) \mapsto f(\alpha)\beta^N (= f(\alpha)/a(\alpha)^N).$$

Exercise 16. Let X be an affine variety with affine coordinate ring $A := A(X)$, let $a \in A$, let U be an open subset of $D_X(a)$. Recall that $\iota_{X,a} : X_a \to X$ is the morphism of affine varieties with image $D_X(a)$ corresponding to the localization-at-a map $\iota_{X,a}^\sharp : A \to A(X_a) = A_a$ of affine coordinate rings. Set $V := \iota_{X,a}^{-1}(U)$. Show that a function $f : U \to k$ belongs to $\mathcal{O}(U)$ if and only if $f|_{X,a} := f \circ \iota_{X,a}$ belongs to $\mathcal{O}(V)$.

8.3 The sheaf of regular functions

The following assertions concerning regular functions are evident (from characterization (R2), for instance):[13]

1. The space $\mathcal{O}(U)$ of regular functions defines a k-subalgebra of the space $\mathrm{Func}(U, k)$ of all functions $U \to k$: every constant function $c \in k$ belongs to $\mathcal{O}(U)$, and both $f_1 + f_2$ and $f_1 f_2$ belong to $\mathcal{O}(U)$ whenever f_1 and f_2 both do.

2. Restrictions of regular functions are regular: if $U \supset U'$ and $f \in \mathcal{O}(U)$, then $f|_{U'} \in \mathcal{O}(U')$.

3. Being a regular function is a local notion: a function $f : U \to k$ belongs to $f \in \mathcal{O}(U)$ whenever there exists an open cover $U = \cup_i U_i$ so that each $f|_{U_i} \in \mathcal{O}(U_i)$.

These conditions imply also that

- restriction is associative in the sense that for $U \supset U' \supset U''$, the composition of restriction maps $\mathcal{O}(U) \to \mathcal{O}(U') \to \mathcal{O}(U'')$ is the same as the restriction map $\mathcal{O}(U) \to \mathcal{O}(U'')$,

[13] The reader is invited to compare with the discussion in Milne's course notes linked on the homepage.

- for $U \supset U'$, the restriction map $\mathcal{O}(U) \to \mathcal{O}(U')$ is a k-algebra morphism, and
- for each open cover $U = \cup_i U_i$ of U with overlaps $U_{ij} := U_i \cap U_j$ and each collection of regular functions $f_i \in \mathcal{O}(U_i)$ satisfying the overlap condition $f_i|_{U_{ij}} = f_j|_{U_{ij}}$, there exists a unique $f \in \mathcal{O}(U)$ for which $f|_{U_i} = f_i$ for all i.

We summarize the above observations with the following definition:

Definition 56. Let X be a topological space. By a k-*sheaf* $\mathcal{O}$ on X we shall mean the association to each open $U \subset X$ of some $\mathcal{O}(U) \subset \mathrm{Func}(U, k)$ satisfying properties 1,2,3 enumerated above.[14]

The association to each open subset U of an affine variety X the space $\mathcal{O}(U)$ of regular functions on U is the prototypical example of a k-sheaf.

Exercise 17. Let
$$X := k[x, y, z, w]/(xw - yz)$$
and
$$U := D_X(x) \cap D_X(z).$$
Verify that the regular functions w/z on $D_X(z)$ and y/x on $D_X(x)$ agree on the overlap, hence induce a regular function $f : U \to k$.

Remark 57. Let X be an affine variety, set $A := A(X)$, and let $U \subset X$ be open. For any basic cover $(X_{a_i} \to U \subset X)_{i \in I}$, the restriction map $\mathcal{O}(U) \ni f \mapsto (f|_{X_{a_i}})_{i \in I}$ induces a natural bijection
$$\mathcal{O}(U) \cong \{(f_i)_{i \in I} \in \prod_{i \in I} A_{a_i} : f_i|_{X_{a_i a_j}} = f_j|_{X_{a_i a_j}}\},$$
giving an explicit description of $\mathcal{O}(U)$ in terms of localizations of the affine coordinate ring of X.

Exercise 18. Let
$$X := \mathbb{A}^2 := \operatorname{Specm} k[x_1, x_2]$$
and
$$U := \mathbb{A}^2 - \{(0, 0)\}.$$
Show that the restriction map
$$A(X) = \mathcal{O}(X) \to \mathcal{O}(U)$$
is an isomorphism. [Apply Remark 57 with the basic cover $(X_{x_i} \to U \subset X)_{i=1,2}$ from Example 49.]

[14] A clunkier but more accurate name than "k-sheaf" might perhaps be "sub-k-algebra-sheaf of $\mathrm{Func}(\cdot, k)$."

9 The category of varieties

9.1 Overview

In this section we define a class of *varieties*, enlarging the class of affine varieties considered thus far, and define morphisms between them in a way that recovers the earlier notion of a polynomial mapping between affine varieties. Our hope is that this discussion makes the definitions given in Hartshorne (of a variety being either quasi-affine or quasi-projective) seem a bit less ad hoc, and also to demonstrate in a very concrete setting some of the ideas involved in the definition of a scheme. We shall be brief; the reader is encouraged to compare the discussion recorded here with that in

- sections II.1 and II.2 of Hartshorne or in ???,
- ??? in Milne,
- ??? in Gathmann,

and perhaps elsewhere.

9.2 Spaces

9.2.1 Definitions

Definition 58. By a k-*space* we shall mean a pair $(X, \mathcal{O}_X)$, where

- X is a topological space, and
- $\mathcal{O}_X$ is a k-sheaf on X (Definition 56).

Intuitively, this definition formalizes the notion of "a space with a local notion of k-valued regular function." We call $\mathcal{O}_X(U)$ the space of *regular functions* on U. We might abbreviate a k-space $(X, \mathcal{O}_X)$ as simply X, with the datum of $\mathcal{O}_X$ implicit, and might also write $\mathcal{O}_X(U)$ as simply $\mathcal{O}(U)$.

Definition 59. A *morphism* of k-spaces $f : Y \to X$ is a continuous function with the property that for each open $U \subset X$ and $g \in \mathcal{O}_X(U)$, the pullback $g \circ f|_{f^{-1}(U)}$ belongs to $\mathcal{O}_Y(f^{-1}(U))$; we thereby obtain k-algebra morphisms

$$f_U^\sharp : \mathcal{O}_X(U) \to \mathcal{O}_Y(f^{-1}(U))$$

via pullback for each open $U \subset X$. The identity map and compositions of morphisms are morphisms, so we obtain in this way a category and a notion of *isomorphism* of k-spaces.

Exercise 19. A function $f : Y \to X$ between k-spaces is an isomorphism if and only if

- f is a homeomorphism, and
- f preserves regularity in both directions, i.e.: for each open $U \subset X$ and function $g \in \operatorname{Func}(U, k)$, one has

$$g \in \mathcal{O}_X(U) \iff g \circ f|_{f^{-1}(U)} \in \mathcal{O}_Y(f^{-1}(U)).$$

9.2.2 Affine varieties are spaces

An affine variety X is naturally a k-space upon taking $\mathcal{O}_X(U) := \mathcal{O}(U)$ to be the space of regular functions as defined before.

Lemma 60. *For affine varieties X, Y, the following sets are the same:*

$$\{ \text{ polynomial maps } Y \to X \ \},$$

$$\{ \text{ morphisms } Y \to X, \text{ regarded as } k\text{-spaces } \}.$$

In particular, two affine varieties are isomorphic as k-spaces if and only if they are isomorphic in the sense defined previously.

Proof. If $f : Y \to X$ is a morphism, then taking $U := X$ and using that $\mathcal{O}(X) = A(X), \mathcal{O}(Y) = A(Y)$ we see that $g \in A(X)$ implies $g \circ f \in A(Y)$, whence f is a polynomial map. Conversely, assume $f : Y \to X$ is a polynomial map. We have seen that f is continuous (Section 6.7). Let U be an open subset of X, and let $g \in \mathcal{O}(U)$. Since g is locally (on U) a ratio of polynomials and f is a polynomial map, we deduce that $g \circ f|_{f^{-1}(U)}$ is locally (on $f^{-1}(U)$) a ratio of polynomials, hence (by (R2)) is regular, as required. Therefore f is a morphism. $\square$

9.2.3 Open subsets of spaces are spaces

An open subset U of a k-space X has the natural structure of a k-space $(U, \mathcal{O}_X|_U)$, where for each open $V \subset U$ we set

$$\mathcal{O}_X|_U(V) := \mathcal{O}_X(V).$$

As for X, we might abbreviate the pair $(U, \mathcal{O}_X|_U)$ as simply U. Given an isomorphism of k-spaces $f : Y \xrightarrow{\cong} X$ and an open subset U of X, the induced map $f : f^{-1}(U) \to U$ is also an isomorphism of k-spaces.

Definition 61. An open subset U of an affine variety X is called a *quasi-affine variety.*

9.3 Prevarieties

Definition 62. A k-space will be called *affine* if it is isomorphic to an affine variety, with the latter regarded as a k-space.

Exercise 20. Let X be an affine variety and $a \in A(X)$. Then $D_X(a)$ is isomorphic to X_a. In particular, $D_X(a)$ is affine. [Combine Exercises 16 and 19.]

Definition 63. By a *prevariety* we shall mean a k-space $X := (X, \mathcal{O}_X)$ for which there is a finite open cover $X = \cup U_i$ with the property that each $(U_i, \mathcal{O}_X|_{U_i})$ is affine.

Example 64. Any open subset U of a prevariety X is a prevariety: Let $\varphi_i : U_i \xrightarrow{\cong} X_i$ be isomorphisms taking U_i to an affine variety X_i, where $X = \cup U_i$ is a finite open cover. Each intersection $U \cap U_i$ maps under φ_i to an open subset $U \cap U_i$, which is a finite union of basic affine open subsets $D_{X_i}(a)$ $(a \in A(X_i))$, each of which is affine (Exercise 20). The inclusion map $U \hookrightarrow X$ is a morphism. In particular, any quasi-affine variety is a prevariety.

Exercise 21. Any regular function on an open subset of a prevariety is continuous.

9.4 Construction via charts

(Not discussed in lecture)

Definition 65. By a *charted prevariety* we shall mean the datum $(X, (U^i \xrightarrow{\varphi_i} X^i)_{i \in I})$, where

- X is a set,
- I is a finite indexing set,
- the U^i are subsets of X whose union is X,
- the X^i are affine varieties, and
- the φ_i are bijective functions (called *charts*)

satisfying some conditions to be enunciated after introducing some additional notation. For each pair i, j, set $U^{ij} := U_i \cap U_j =: U^{ji}$ and $X^{ij} := \varphi_i(U^{ij}) \subset X^i$, so that $X^{ji} = \varphi_j(U^{ij}) \subset X^j$, allowing us to define a bijection $\varphi_{ij} := \varphi_j \circ \varphi_i^{-1} : X^{ij} \to X^{ji}$. We then impose the following conditions:

1. Each set X^{ij} is open in X^i, hence may be regarded as a quasi-affine variety.
2. Each map φ_{ij} is a morphism of quasi-affine varieties.

Since $\varphi_{ji} \circ \varphi_{ij} = 1_{X^{ij}}$, it follows immediately that each φ_{ij} is an isomorphism. These conditions allow us to define:

- a *topology* on X, by requiring that each chart φ_i be a homeomorphism;
- for each open subset $U \subset X$, a space $\mathcal{O}_X(U)$ of *regular functions* $f : U \to k$ characterized by requiring that f induce a regular function on each chart, i.e., that for each i the composition

  $$f \circ \varphi_i^{-1} : \varphi_i(U) \cap X^i \to k$$

  define a regular function on the quasi-affine variety $\varphi_i(U) \cap X^i$.

Exercise 22. Verify that for each charted prevariety $(X, (X^i \to U^i)_i)$, the pair $(X, \mathcal{O}_X)$ obtained by "forgetting the charts" defines a prevariety. Conversely, every prevariety X arises in this way (with $(U^i)_{i \in I}$ any finite open cover of X). For two prevarieties $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ arising in this way from charted prevarieties $(X, (U^i \xrightarrow{\phi_i} X^i)_i)$ and $(Y, (V^j \xrightarrow{\psi_j} Y^j)_j)$, verify that morphisms $Y \to X$ are "determined chart-by-chart" in the sense that a function $f : Y \to X$ defines a morphism of prevarieties if and only if for each i, j, the induced map

$$Y^j \cap \psi_j(f^{-1}(U^i)) \xrightarrow{\cong} V^j \cap f^{-1}(U^i) \xrightarrow{f} U^i \xrightarrow{\cong} X^i$$

is a morphism of quasi-affine varieties.

Example 66. Let X be a quasi-affine variety with closure the affine variety $\overline{X}$. Let

$$(\overline{X}_{a_i} \to X \subset \overline{X})_{i \in I}$$

be a basic finite cover. Take $U^i := D_{\overline{X}}(a_i)$, $X^i := \overline{X}_{a_i}$, and $\varphi_i := \iota_{\overline{X}, a_i}$. Then φ_{ij} is the composition $X^{ij} = D_{\overline{X}_{a_i}}(a_j) \xrightarrow{\cong} \overline{X}_{a_i a_j} \xrightarrow{\cong} D_{\overline{X}_{a_j}}(a_i) = X^{ji}$ and $(X, (U^i \xrightarrow{\varphi_i} X^i)_{i \in I})$ is a charted prevariety. Since a function $f : U \to k$ on an open subset U of X is regular if and only if each $f|_{U \cap U_i}$ is regular if and only if each $f \circ \varphi_i^{-1}|_{\varphi(U \cap U_i)}$ is regular, we see that the prevariety $(X, \mathcal{O}_X)$ obtained directly from the quasi-affine variety X coincides with that obtained from the charted prevariety $(X, (U^i \xrightarrow{\varphi_i} X^i)_{i \in I})$.

9.5 Varieties

Definition 67. A prevariety X will be called *separated* if for each prevariety Z and each pair of morphisms $f_1, f_2 : Z \to X$, the subset

$$\mathrm{eq}(f_1, f_2) := \{\gamma \in Z : f_1(\gamma) = f_2(\gamma)\}$$

of Z is closed. By a *variety* we shall mean a separated prevariety.

Remark 68. The notion of a separated prevariety is the appropriate analogue of the notion of a Hausdorff topological space. Indeed, a topological space X is Hausdorff if and only if the diagonal ΔX is closed in the product space $X \times X$: to compare with the more customary definition, a basis of neighborhoods near $(x, y) \in X \times X$ is given by the products $U \times V$ of pairs of neighborhoods $x \in U \subset X$, $y \in V \subset Y$, which satisfy $U \times V \cap \Delta X = \emptyset$ iff $U \cap V = \emptyset$. On the other hand, ΔX is the set where the projection maps $p_1, p_2 : X \times X \to X$ coincide, and the set $\mathrm{eq}(f_1, f_2)$ as defined above where a pair of continuous functions $f_1, f_2 : Z \to X$ from a topological space Z coincide is the preimage of ΔX under the product morphism $f_1 \times f_2 : Z \to X \times X$. Thus a topological space X is Hausdorff if and only if for each topological space Z and pair of continuous maps $f_1, f_2 : Z \to X$, the subset $\mathrm{eq}(f_1, f_2)$ defined as above is closed in Z.

Exercise 23. The definition of "separated" remains unchanged if one restricts Z to be an affine variety.

Example 69. Any quasi-affine variety

$$X \subset \mathbb{A}^n := \operatorname{Specm} k[x_1, \ldots, x_n]$$

is separated, hence a variety: For $f_1, f_2 : Z \to X$ as in Definition 63, the set $\operatorname{eq}(f_1, f_2)$ is the intersection over all coordinate indices $j \in \{1, \ldots, n\}$ of the sets

$$\{\gamma \in Z : \overline{x}_j(f_1(\gamma)) = \overline{x}_j(f_2(\gamma))\}, \quad \overline{x}_j := x_j|_X,$$

which are the preimages of $\{0\}$ under the regular functions $\overline{x}_j \circ f_1 - \overline{x}_j \circ f_2$ and hence closed thanks to the continuity of the latter.[15]

Example 70 (The affine line with a doubled point)**.** (Not discussed in lecture.) Let X be the set

$$(\mathbb{A}^1 - \{0\}) \sqcup \{*_1\} \sqcup \{*_2\},$$

$I := \{1, 2\}$ $U^j := (k - \{0\}) \sqcup \{*_j\}$, $X^j := \mathbb{A}^1$, and $\varphi_j : X^j \to U^j$ given by

$$\varphi_j(\gamma) := \begin{cases} \gamma & \text{if } \gamma \neq 0 \\ \{*_j\} & \text{if } \gamma = 0. \end{cases}$$

Then the prevariety $(X, \mathcal{O})$ obtained from the charted prevariety $(X, (U^i \xrightarrow{\varphi_i} X^i)_{i \in I})$ is not separated, since $\operatorname{eq}(\varphi_1, \varphi_2) = \mathbb{A}^1 - \{0\}$ is not closed. (See for instance Example II.2.3.6 and the discussion on p97 of Hartshorne.)

Example 71. Let X be a variety, and let $U \subset X$ be an open subset. Then U is a variety: We saw in 64 that it is a prevariety, and it is also separated: any morphisms $f_1, f_2 : Z \to U$ compose with the inclusion morphism $U \to X$ to give morphisms $h_1, h_2 : Z \to X$ for which $\operatorname{eq}(h_1, h_2) = \operatorname{eq}(f_1, f_2)$ is closed by the assumption that X is separated.

9.6 Complements

Exercise 24. Show that the quasi-affine variety $U := \mathbb{A}^2 - \{(0, 0)\}$ is not isomorphic to any affine variety. [Use Exercise 49 and Lemma 60.]

Exercise 25 (Morphisms of quasi-affine in terms of polynomial maps between affine varieties)**.** Let X, Y be quasi-affine varieties with closures $\overline{X}, \overline{Y}$. Let $f : Y \to X$ be any function. Show that the following are equivalent:

1. For each $a \in A(\overline{X})$ with $U := D_{\overline{X}}(a) \subset X$, the preimage $f^{-1}(U)$ is open and for each basic open $D_{\overline{Y}}(b) \subset f^{-1}(U)$, the composition $\overline{Y}_b \cong D_{\overline{Y}}(b) \xrightarrow{f} U \cong \overline{X}_a$ is a polynomial map between affine varieties.

[15] There are "better" ways to show this, discussed in virtually every reference, that I don't plan to spend time on; see for instance Hartshorne, Section II.4 or the relevant sections of the notes by Milne or Gathmann.

2. f is a morphism.

3. There exists a basic cover $(\overline{X}_{a_i} \to X \subset \overline{X})_{i \in I}$ so that for each $i \in I$, the preimage $f^{-1}(U)$ of $U := D_{\overline{X}}(a_i)$ is open and there exists a basic cover $(\overline{Y}_{b_j} \to f^{-1}(U) \subset \overline{Y})_{j \in J}$ so that for each $j \in J$, the composition $\overline{Y}_{b_j} \cong D_{\overline{Y}}(b_j) \xrightarrow{f} U \cong \overline{X}_{a_i}$ is a polynomial map between affine varieties.

[Use the equivalences of the characterizations (R1), (R2), (R3) in the definition of a regular map.]

Exercise 26.

1. Verify for a prevariety X that morphisms $X \to \mathbb{A}^1$ are the same as regular functions $X \to k$.

2. Verify for an *affine* variety X and a prevariety Y that morphisms of prevarieties
$$f : Y \to X$$
are in bijection with k-algebra morphisms
$$\phi : A(X) \to \mathcal{O}(Y)$$
under the mutually inverse maps $f \mapsto f^\sharp$ and $\phi \mapsto \phi^\flat$, where $f^\sharp(a) := a \circ f$ and $\phi^\flat(\beta) := \mathrm{pt}(\mathrm{eval}_\beta \circ \phi)$ for $a \in A(X), \beta \in Y$.

3. Explain how the second part of this exercise specializes to the first part.

Exercise 27. Let $(X, \mathcal{O}_X)$ be a prevariety and $\alpha \in X$. Verify the following assertions:

1. The neighborhoods U of α form a directed set with respect to containment. The *stalk of* $\mathcal{O}_X$ *at* α is defined to be
$$\mathcal{O}_{X,\alpha} := \varinjlim_{U \ni \alpha} \mathcal{O}_X(U).$$
It is naturally a k-algebra.

2. For any neighborhood U of α, the evaluation-at-α morphism $\mathrm{eval}_\alpha : \mathcal{O}_X(U) \to k$ induces a morphism $\mathcal{O}_{X,\alpha} \to k$. We denote the latter also by eval_α.

3. The k-algebra $\mathcal{O}_{X,\alpha}$ is a local ring. Its unique maximal ideal is the kernel of $\mathrm{eval}_\alpha : \mathcal{O}_{X,\alpha} \to k$.

4. For any affine neighborhood Y of α, the open neighborhoods $D_Y(a)$ of α (taken over $a \in A(Y)$ with $a(\alpha) \neq 0$) form a directed set that is cofinal in the the directed set of all neighborhoods of α. The restriction maps $\mathcal{O}_X(U) \to \mathcal{O}_X(D_Y(a))$ for $\alpha \in D_Y(a) \subset U$ thereby induce an identification
$$\mathcal{O}_{X,\alpha} = \varinjlim_{D_Y(a) \ni \alpha} \mathcal{O}_X(D_Y(a)) = \varinjlim_{D_Y(a) \ni \alpha} A(Y)_a \cong A(Y)_{\mathfrak{m}}$$
where $\mathfrak{m}$ is the maximal ideal $\mathfrak{m} := \{a \in A(Y) : a(\alpha) = 0\}$ of $A(Y)$.

Exercise 28. Verify that the affine line with doubled point from Example 70 is homeomorphic to the affine line $\mathbb{A}^1$. Deduce that there does not a characterization of when a prevariety $(X, \mathcal{O}_X)$ is separated that depends only upon the underlying topological space X.

10 Projective varieties: basics

10.1 Overview

We aim here to introduce projective varieties and their basic properties by considering what happens when we take the basic definitions and results concerning affine varieties inside

$$\mathbb{A}^{n+1} := \operatorname{Specm} k[x], \quad k[x] := k[x_0, \ldots, x_n]$$

and consider how they interact with the *dilation action* by $k^\times$, where each $\lambda \in k^\times$ sends a point $\alpha = (\alpha_0, \ldots, \alpha_n) \in \mathbb{A}^{n+1}$ to $\lambda\alpha := (\lambda\alpha_0, \ldots, \lambda\alpha_n)$.

10.2 Projective space

The set of orbits in $\mathbb{A}^{n+1} - \{0\}$ for this action is called *projective n-space* and denoted $\mathbb{P}^n$. The class of a point

$$\alpha = (\alpha_0, \ldots, \alpha_n) \in \mathbb{A}^{n+1} - \{0\}$$

is denoted

$$[\alpha] = [\alpha_0, \ldots, \alpha_n] \in \mathbb{P}^n.$$

Every element of $\mathbb{P}^n$ is thus of the form $[\alpha]$ for some $\alpha \in \mathbb{A}^{n+1} - \{0\}$ and two such classes $[\alpha], [\beta]$ are the same if and only if there exists $\lambda \in k^\times$ for which $\beta = \lambda\alpha$. There is a natural surjective map

$$\pi : \mathbb{A}^{n+1} - \{0\} \to \mathbb{P}^n$$

$$\alpha \mapsto [\alpha]$$

for which the fiber above $[\alpha]$ is the set $\{\lambda\alpha : \lambda \in k^\times\}$, which is the complement of $\{0\}$ in the line containing 0 and α.

It is also profitable to identify $\mathbb{P}^n$ with the space of lines in $\mathbb{A}^{n+1}$ containing 0, where to each $[\alpha] \in \mathbb{P}^n$ corresponds the line

$$0 \in \ell_{[\alpha]} \subset \mathbb{A}^{n+1}$$

defined by

$$\ell_{[\alpha]} := \{0\} \cup \pi^{-1}([\alpha]) = \{\lambda\alpha : \lambda \in k\}.$$

10.3 Homogeneous polynomials

Consider now the action induced by dilation on polynomials. Given a polynomial $f \in k[x]$, its value at the dilate $\lambda\alpha$ of some $\alpha \in \mathbb{A}^{n+1}$ may be written

$$f(\lambda\alpha) = \sum_{d \in \mathbf{Z}_{\geq 0}} \lambda^d f_d(\alpha), \tag{12}$$

where the f_d are the polynomials obtained from f by taking the sum of the terms of degree d. We obtain in this way a grading

$$k[x] = \oplus_{d \in \mathbf{Z}_{\geq 0}} k[x]_d.$$

A polynomial belonging to some $k[x]_d$, such as

$$f = x_0^2 - x_1^2 - x_2^2 \in k[x_0, x_1, x_2]_2,$$

is called *homogeneous*; equivalently, $f \in k[x]$ is homogeneous if it is a sum of monomials of the same degree. The polynomial $f_d \in k[x]_d$ attached to $f \in k[x]$ as above is called the *dth homogeneous component* of f.

For $f \in k[x]_d$, one has

$$f(\lambda\alpha) = \lambda^d f(\alpha)$$

for all α, λ as above. In other words, a homogeneous polynomial of degree d is an *eigenfunction* for the dilation action by $\lambda \in k^\times$ with eigenvalue given by the character $\lambda \mapsto \lambda^d$.

In summary, the decomposition of a polynomial into homogeneous components is closely related to the dilation action of $k^\times$.

10.4 Notions that are well-defined

10.4.1 The vanishing (or not) of a homogeneous polynomial at a point of projective space

The value of a polynomial $f \in k[x]$ at an equivalence class $[\alpha] \in \mathbb{P}^n$ is not, in general, well-defined: it might very well be the case that $f(\alpha) \neq f(\lambda\alpha)$ (and yet of course $[\alpha] = [\lambda\alpha]$) for some $\lambda \in k^\times$. However, when f is homogeneous, that is to say, belongs to $k[x]_d$ for some $d \in \mathbf{Z}_{\geq 0}$, the *truth value* of the statement

$$f([\alpha]) = 0$$

is well-defined, because $f(\alpha) = 0$ if and only if $f(\lambda\alpha) = 0$ for any $\lambda \in k^\times$. Similarly, the truth value of

$$f([\alpha]) \neq 0$$

is well-defined.

10.4.2 The ratio of two homogeneous polynomials of the same degree

When g, h are homogeneous polynomials of the same degree, that is to say, $g, h \in k[x]_d$ for some $d \in \mathbf{Z}_{\geq 0}$, and $[\alpha] \in \mathbb{P}^n$ is a point for which $h([\alpha]) \neq 0$, the ratio

$$\frac{g}{h}([\alpha]) := \frac{g(\alpha)}{h(\alpha)}$$

is well-defined, since for any $\lambda \in k^\times$,

$$\frac{g(\lambda\alpha)}{h(\lambda\alpha)} = \frac{\lambda^d g(\alpha)}{\lambda^d h(\alpha)} = \frac{g(\alpha)}{h(\alpha)}.$$

10.5 Zariski topology

For a homogeneous $f \in k[x]$, the subset

$$D_{\mathbb{P}^n}(f) := \{[\alpha] \in \mathbb{P}^n : f([\alpha]) \neq 0\}$$

is well-defined. For most of this section we shall abbreviate

$$D(f) := D_{\mathbb{P}^n}(f),$$

taking care not to confuse that set with what we shall denote by

$$D_{\mathbb{A}^{n+1}}(f) := \{\alpha \in \mathbb{A}^{n+1} : f(\alpha) \neq 0\}.$$

Note that

$$D(f_1) \cap D(f_2) = D(f_1 f_2).$$

The *Zariski topology* on $\mathbb{P}^n$ is defined to be the topology generated by the $D(f)$ as an open basis. In other words, a subset of $\mathbb{P}^n$ is Zariski open if and only if it is a union of sets of the form $D(f)$. Note that

10.6 Definition of projective varieties

A projective variety $X \subset \mathbb{P}^n$ is defined to be the solution set to a system of homogeneous equations, i.e., $X = V_{\mathbb{P}^n}(S)$, where

$$V_{\mathbb{P}^n}(S) := \{[\alpha] \in \mathbb{P}^n : f([\alpha]) = 0 \text{ for all homogeneous } f \in S\},$$

for some subset $S \subset k[x]$. In practice we shall only write $V_{\mathbb{P}^n}(S)$ when the set S is itself *homogeneous* in the sense that it contains the homogeneous components of each of its elements.

We shall occasionally abbreviate

$$V(S) := V_{\mathbb{P}^n}(S)$$

in this section, taking care to distinguish that set from the affine variety

$$V_{\mathbb{A}^{n+1}}(S) := \{\alpha \in \mathbb{A}^{n+1} : f(\alpha) = 0 \text{ for all } f \in S\}.$$

The following are evidently equivalent:

- X is a projective variety.
- X is an intersection of hypersurfaces $V(f) := V(\{f\}) := V_{\mathbb{P}^n}(\{f\})$ ($f \in k[x]$, homogeneous).
- The complement of X is a union of basic opens $D(f)$ ($f \in k[x]$, homogeneous).
- The complement of X is open.
- X is closed.

We take on each projective variety X the induced topology, or equivalently, the topology with open basis given by the basic open sets

$$D_X(f) := \{[\alpha] \in X : f([\alpha]) \neq 0\}$$

for f homogeneous.

10.7 Affine coordinate patches

10.7.1 Motivating observations

We make the following observations about projective space $\mathbb{P}^n$:

- For each $[\alpha] \in \mathbb{P}^n$ there exists a coordinate index i for which $\alpha_i \neq 0$, that is to say,
$$\mathbb{P}^n = \cup_{i=0}^n D(x_i).$$
- For each $[\alpha] \in D(x_i)$ the corresponding line $\ell_{[\alpha]} \subset \mathbb{A}^{n+1}$ meets the hyperplane defined by $x_i = 1$ in exactly one point, namely
$$\ell_{[\alpha]} \cap \{(\alpha_0, \ldots, \alpha_n) \in \mathbb{A}^{n+1} : \alpha_i = 1\} = (\alpha_0/\alpha_i, \ldots, \alpha_n/\alpha_i).$$
- We thus obtain a natural bijection between $D(x_i)$ and a hyperplane. The latter is an affine variety isomorphic to $\mathbb{A}^n$, and so we obtain in this way a cover of $\mathbb{P}^n$ by the $(n+1)$ subsets $D(x_i)$ each of which carry the structure of an affine variety isomorphic to $\mathbb{A}^n$.

(The reader is encouraged to draw pictures illustrating the above discussion in the special cases $n = 1, 2$.)

10.7.2 Some notation

We set some notation here that will help us keep track of certain variable substitutions. For $i \in \{0..n\}$, consider the $(n+1)$-dimensional affine space

$$\mathbb{A}^{n+1} = \operatorname{Specm} k[x_{0/i}, \ldots, x_{n/i}]$$

with the formal coordinate variables $x_{0/i}, \ldots, x_{n/i}$. Denote by $\mathbb{P}^n_i$ the hyperplane cut out by requiring that the ith coordinate equal one, i.e.,

$$\mathbb{P}^n_i := \left\{(\alpha_{0/i}, \ldots, \alpha_{n/i}) \in \mathbb{A}^{n+1} : \alpha_{i/i} = 1\right\}.$$

We picture $\mathbb{P}^n_i$ as the hyperplane $\{x_i = 1\}$ discussed in Section 10.7.1, but with coordinates variables $x_{0/i}, \ldots, x_{n/i}$ instead of $x_0, \ldots, x_n$. The corresponding affine coordinate ring is

$$R_i := A(\mathbb{P}^n_i) = \frac{k[x_{0/i}, \ldots, x_{n/i}]}{(x_{i/i} - 1)}.$$

The natural map

$$\mathbb{A}^n = \operatorname{Specm} k[x_{0/i}, \ldots, \widehat{x_{i/i}}, \ldots, x_{n/i}] \to \mathbb{P}^n_i$$

$$(\alpha_{0/i}, \ldots, \widehat{\alpha_{i/i}}, \ldots, \alpha_{n/i}) \mapsto (\alpha_{0/i}, \ldots, 1, \ldots, \alpha_{n/i}),$$

where $\widehat{\cdots}$ denotes an omitted variable, is an isomorphism of affine variables, corresponding on coordinate rings to the isomorphism

$$R_i \xrightarrow{\cong} k[x_{0/i}, \ldots, \widehat{x_{i/i}}, \ldots, x_{n/i}]$$

induced by "setting $x_{i/i} := 1$."

10.7.3 Charts

The "send $[\alpha]$ to the intersection of the line $\ell_{[\alpha]}$ with the hyperplane $\{x_i = 1\}$" map discussed in Section 10.7.1 translates with coordinate system introduced above to the map

$$\varphi_i : D(x_i) \to \mathbb{P}^n_i$$

given by

$$[\alpha_0, \ldots, \alpha_n] \mapsto \left(\frac{\alpha_0}{\alpha_i}, \ldots, \frac{\alpha_n}{\alpha_i}\right).$$

The inverse map, described geometrically as "send a point on the hyperplane $\{x_i = 1\}$ to the line through 0 containing it," translates to

$$\varphi_i^{-1} : \mathbb{P}^n_i \to D(x_i)$$

$$(\alpha_{0/i}, \ldots, \alpha_{n/i}) \mapsto [\alpha_{0/i}, \ldots, \alpha_{n/i}].$$

The geometric description makes clear that these two maps φ_i and φ_i^{-1} are mutually inverse bijections, justifying the notation. That this is so may be alternatively verified by performing the calculations

$$(\alpha_{0/i}, \ldots, \alpha_{n/i}) \mapsto [\alpha_{0/i}, \ldots, \alpha_{n/i}] \mapsto \left(\frac{\alpha_{0/i}}{\alpha_{i/i}}, \ldots, \frac{\alpha_{n/i}}{\alpha_{i/i}}\right) = (\alpha_{0/i}, \ldots, \alpha_{n/i})$$

and

$$[\alpha_0, \ldots, \alpha_n] \mapsto \left(\frac{\alpha_0}{\alpha_i}, \ldots, \frac{\alpha_n}{\alpha_i}\right) \mapsto \left[\frac{\alpha_0}{\alpha_i}, \ldots, \frac{\alpha_n}{\alpha_i}\right] = [\alpha_0, \ldots, \alpha_n]$$

which follow from the definition of projective space $\mathbb{P}^n$ and the requirement $\alpha_{i/i} = 1$ in the definition of the hyperplane $\mathbb{P}^n_i$.

As motivation for the choice of notation, note that the regular function x_j/x_i on $D(x_i)$ pulls back under φ_i^{-1} to the regular function $x_{j/i}$ on $\mathbb{P}^n_i$, and vice versa:

$$x_{j/i} \circ \varphi_i = \frac{x_j}{x_i} \text{ as regular functions on } D(x_i),$$

$$x_{j/i} = \frac{x_j}{x_i} \circ \varphi_i^{-1} \text{ as regular functions on } \mathbb{P}^n_i.$$

10.8 Homogenization

10.8.1 Definition

Let $f \in R_i$ be a regular function on $\mathbb{P}^n_i$, such as (for instance) the polynomial

$$f := x_{2/0}^2 - (x_{1/0}^3 - x_{1/0}) \in R_0$$

on $\mathbb{P}^2_0$. One may equivalently regard f as a polynomial in the n formal variables $x_{0/i}, \ldots, \widehat{x_{i/i}}, \ldots, x_{n/i}$ or as a polynomial in the $n+1$ variables $x_{0/i}, \ldots, x_{n/i}$ but with the variable $x_{i/i}$ (and only that variable) treated as a "dummy variable" satisfying $x_{i/i} := 1$.

The process of *homogenization* mentioned in the title consists of attaching to f a *homogeneous* polynomial $f^* \in k[x]$ that "looks like f on the hyperplane $\{x_i = 1\}$." Mechanically, this is effected by replacing each occurrence of the variable $x_{j/i}$ in f with the ratio x_j/x_i and then multiplying by a large enough positive power of x_i to clear denominators. The definition we shall adopt is

$$f^*(x_0, \ldots, x_n) := x_i^{\deg(f)} f\left(\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}\right),$$

where $\deg(f)$ is the maximum of the degrees of the monomials occurring in the (not necessarily homogeneous) polynomial f, or equivalently, the smallest integer for which the RHS of the above defines a polynomial function of $x_0, \ldots, x_n$. In other words, one gets from f to f^* by replacing each $x_{j/i}$ with x_j and then multiplying each monomial term by a suitable power of x_i to make it homogeneous of degree $\deg(f)$.

Example 72. Suppose that $n = 2$, $i = 0$, and

$$f = y^2 - (x^3 - x) \in R_0$$

with the shorthand $x := x_{1/0}, y := x_{2/0}$. Then

$$f^*(x_0, x_1, x_2) = x_0^3 f\left(\frac{x_0}{x_0}, \frac{x_1}{x_0}, \frac{x_2}{x_0}\right) = x_0 x_2^2 - (x_1^3 - x_0^2 x_1).$$

Example 73. Suppose again that $n = 2$, $i = 0$, but now that

$$f = 1 - (x^2 + y^2) \in R_0,$$

again with the shorthand $x := x_{1/0}, y := x_{2/0}$. Then

$$f^*(x_0, x_1, x_2) = x_0^2 f\left(\frac{x_0}{x_0}, \frac{x_1}{x_0}, \frac{x_2}{x_0}\right) = x_0^2 - x_1^2 - x_2^2.$$

10.8.2 Functional properties

As a function, f^* may be defined in terms of f by requiring that any of the following evidently equivalent conditions are satisfied:

- $f^*(\alpha_{0/i}, \ldots, \alpha_{n/i}) = f(\alpha_{0/i}, \ldots, \alpha_{n/i})$ for all $(\alpha_{0/i}, \ldots, \alpha_{n/i}) \in \mathbb{P}_i^n$.
- $f^*(\alpha_0, \ldots, \alpha_n)/\alpha_i^d = f\left(\frac{\alpha_0}{\alpha_i}, \ldots, \frac{\alpha_n}{\alpha_i}\right)$ for all $(\alpha_0, \ldots, \alpha_n) \in \mathbb{A}^{n+1}$ with $\alpha_i \neq 0$, where d denotes the degree of f^*.
- $\frac{f^*}{x_i^d}([\alpha]) = f(\varphi_i([\alpha]))$ for all $[\alpha] \in D(x_i)$.
- As functions on the open subset $D(x_i)$ of projective space,
$$\frac{f^*}{x_i^d} = f \circ \varphi_i.$$
- As functions on the affine chart $\mathbb{P}_i^n$,
$$\frac{f^*}{x_i^d} \circ \varphi_i^{-1} = f.$$

10.8.3 Compatibility with taking ratios

For any $g, h \in R_i$, the pullback of g/h under φ_i is (where defined) a ratio of homogeneous polynomials of the same degree, namely

$$\frac{g}{h} \circ \varphi_i = \frac{x_i^{\deg(h^*)} g^*}{x_i^{\deg(g^*)} h^*}.$$

10.9 Dehomogenization

10.9.1 Definition

The dehomogenization of a homogeneous polynomial $f \in k[x]$ is defined to be its image $f_{*i} \in R_i$ under the natural map

$$k[x] \to R_i$$

induced by

$$x_j \mapsto x_{j/i},$$

thus

$$f_{*i}(x_{0/i},\dots,x_{n/i}) = f(x_{0/i},\dots,x_{n/i}).$$

Mechanically, the association $f \mapsto f_{*i}$ is carried out by replacing each occurrence of x_j with $x_{j/i}$ and then setting $x_{i/i} = 1$. We picture f_{*i} as the restriction to the hyperplane $\{x_i = 1\}$ of f, but with renamed coordinate functions to avoid confusion. Note that f_{*i} is no longer necessarily homogeneous.

Example 74. Take $n := 3, i := 0$ and

$$f := x_0x_1 - x_2x_3.$$

Then with the shorthand

$$x := x_{1/0}, \quad y := x_{2/0}, \quad z := x_{3/0},$$

one has

$$f_{*0} = x - yz.$$

Example 75. Take $n := 1$, $i := 0$, and

$$f := x_0^2 - x_0x_1.$$

Then

$$\begin{aligned} f_{*0} &= 1 - x_{1/0}, \\ (f_{*0})^* &= x_0 - x_1, \\ f_{*1} &= x_{0/1}^2 - x_{0/1}, \\ (f_{*1})^* &= x_0^2 - x_0x_1. \end{aligned}$$

10.9.2 Functional properties

In terms of functions, we have the following evidently equivalent conditions:

- $f_{*i}(\alpha_{0/i},\dots,\alpha_{n/i}) = f(\alpha_{0/i},\dots,\alpha_{n/i})$ for all $(\alpha_{0/i},\dots,\alpha_{n/i}) \in \mathbb{P}_i^n$.
- $f(\alpha_0,\dots,\alpha_n)/\alpha_i^{\deg(f)} = f_{*i}\left(\frac{\alpha_0}{\alpha_i},\dots,\frac{\alpha_n}{\alpha_i}\right)$ for all $(\alpha_0,\dots,\alpha_n) \in \mathbb{A}^{n+1}$ with $\alpha_i \neq 0$.
- $\frac{f}{x_i^{\deg(f)}}([\alpha]) = f_{*i}(\varphi_i([\alpha]))$ for all $[\alpha] \in D(x_i)$.
- As functions on the open subset $D(x_i)$ of projective space,

$$\frac{f}{x_i^{\deg(f)}} = f_{*i} \circ \varphi_i.$$

- As functions on the affine chart $\mathbb{P}_i^n$,

$$\frac{f}{x_i^{\deg(f)}} \circ \varphi_i^{-1} = f_{*i}.$$

10.9.3 Compatibility with taking ratios

For homogeneous $g, h \in k[x]$ of the same degree, it follows that

$$\frac{g}{h} \circ \varphi_i^{-1} = \frac{g_i}{h_i}.$$

10.9.4 Relationship with homogenization

Dehomogenization is very nearly an inverse to homogenization. Indeed, for any $f \in R_i$ we have

$$(f^*)_{*i} = f.$$

On the other hand, it can happen (as in Example 75) for $f \in k[x]$ that $(f_{*i})^*$ differs from f by some nonzero integral power of x_i. But since x_i and its powers take the value 1 on the hyperplane $\{x_i = 1\}$, such distinction is immaterial in practice. More precisely, there exists $k \in \mathbf{Z}_{\geq 0}$ so that

$$x_i^k (f_{*i})^* = f,$$

and k may be computed explicitly as the largest exponent for which x_i^k divides f. In particular, we always have that

$$(f_{*i})^* \text{ divides } f.$$

10.10 The standard affine charts on projective space are homeomorphisms

Having introduced the requisite notation, we turn now to verifying that the bijective maps $\varphi_i : D(x_i) \to \mathbb{P}_i^n$ introduced in Section 10.7.2 are in fact *homeomorphisms*. We must check that φ_i and φ_i^{-1} map open sets to open sets. It suffices to show that basic open sets are mapped to open sets. We shall show in fact that basic open sets are mapped to basic open sets. In the forward direction, it follows from the discussion of Section 10.9 that for $f \in k[x]$,

$$\varphi_i(D(x_i) \cap D(f)) = D_{\mathbb{P}_i^n}(f_{*i}).$$

Similarly, it follows from the discussion of Section 10.8 that for $f \in R_i$,

$$\varphi_i^{-1}(D_{\mathbb{P}_i^n}(f)) = D(x_i) \cap D(f^*).$$

10.11 Definition of the space of regular functions on a projective variety

Definition 76. Let $X \subset \mathbb{P}^n$ be a projective variety. For each open $U \subset X$, the set

$$\mathcal{O}(U) \subset \mathrm{Func}(U, k)$$

of *regular functions* is defined to consist of those $f : U \to k$ which are locally ratios of homogeneous polynomials of the same degree, that is to say, for which at each point $[\alpha] \in U$ there exists a neighborhood

$$[\alpha] \in V \subset U$$

and homogeneous polynomials $g, h \in k[x]_d$ of the same degree $d \in \mathbf{Z}_{\geq 0}$ so that

- $V \subset D(h)$, i.e., $h([\alpha]) \neq 0$ for all $[\alpha] \in V$, and
- on V, the identity $f = g/h$ holds.

Note that by the discussion of Section 10.4, the ratio g/h is well-defined on V.

We have thus defined $\mathcal{O}(U)$ by an analogue of the condition (R2) used in the case of a quasi-affine variety. By the local nature of the definition of a regular function, it is evident that $U \mapsto \mathcal{O}(U)$ defines a k-sheaf on X (Definition 56) and hence that $(X, \mathcal{O})$ defines a k-space (Definition 58).

10.12 The affine cone over a subset of projective space

Recall the canonical projection $\pi : \mathbb{A}^{n+1} - \{0\} \to \mathbb{P}^n$. The *affine cone* C_X of a subset $X \subset \mathbb{P}^n$ is defined to be

$$C_X := \{0\} \cup \pi^{-1}(X).$$

It is worth drawing a picture of C_X in a special case such as

$$X = V_{\mathbb{P}^2}(x_0^2 - x_1^2 - x_2^2) \subset \mathbb{P}^2$$

for which

$$C_X = V_{\mathbb{A}^3}(x_0^2 - x_1^2 - x_2^2) \subset \mathbb{A}^3.$$

The basic first step towards the local study of a projective variety is to consider the intersections of its affine cone with hyperplanes. In the example just mentioned, these intersections are the *conic sections* familiar from elementary algebra. (Several pictures were attempted in class.)

10.13 Projective varieties are prevarieties, i.e., admit finite affine open covers

10.13.1 The open cover of a projective variety obtained by intersecting its affine cone with hyperplanes

Let $X \subset \mathbb{P}^n$ be a projective variety. Define the index set $I := \{0, 1, \dots, n\}$ and for each $i \in I$, set

$$U_i := D_X(x_i),$$

which we recall for convenience is defined by $D_X(x_i) := \{[\alpha] \in X : \alpha_i \neq 0\} \subset D(x_i) := \{[\alpha] \in \mathbb{P}^n : \alpha_i \neq 0\}$. Set

$$X_i := \varphi_i(U_i) \subset \mathbb{P}^n_i.$$

Thus X_i is essentially the intersection inside $\mathbb{A}^{n+1}$ of the affine cone C_X of X with the hyperplane $\{x_i = 1\}$: if we identify the ambient copies of $\mathbb{A}^{n+1}$, then

$$X_i = C_X \cap \{x_i = 1\}.$$

(In lecture, some pictures were attempted involving $X = V(x_0^2 - x_1^2 - x_2^2)$.)

Example 77. Take $X = V(x_0^2 - x_1^2 - x_2^2) \subset \mathbb{P}^2$. Then

- $X_0 = V(1 - x_{1/0}^2 - x_{2/0}^2)$ is a circle,
- $X_1 = V(x_{0/1}^2 - 1 - x_{2/1}^2)$ is a hyperbola, and
- $X_2 = V(x_{0/2}^2 - x_{1/2}^2 - 1)$ is again a hyperbola.

Now take $Y = V(x_0 x_2^2 - x_1^3 + x_0^2 x_1) \subset \mathbb{P}^2$. Then

- $Y_0 = V(x_{2/0}^2 - x_{1/0}^3 + x_{1/0})$,
- $Y_1 = V(x_{0/1} x_{2/1}^2 - 1 + x_{0/1}^2)$,
- $Y_2 = V(x_{0/2} - x_{1/2}^3 + x_{0/2}^2 x_{1/2})$.

For notational clarity it can be convenient to introduce shorthand such as $x := x_{1/0}, y := x_{2/0}$. One thereby identifies $\mathbb{P}_0^2$ with $\mathbb{A}^2 = \operatorname{Specm} k[x, y]$. Under this identification,

$$X_0 = V(1 - x^2 - y^2), \quad Y_0 = V(y^2 - x^3 + x).$$

10.13.2 The two k-space structures to be compared

Since φ_i is a homeomorphism (Section 10.10), we know that X_i is a closed subset of the affine variety $\mathbb{P}_i^n \cong \mathbb{A}^n$, hence X_i is itself an affine variety. It thus comes equipped with a k-sheaf of regular functions $\mathcal{O}_{X_i}$, and so may be regarded as a k-space

$$(X_i, \mathcal{O}_{X_i}).$$

On the other hand, regarding the projective variety X as a k-space $(X, \mathcal{O})$, its open subset U_i has the induced k-space structure

$$(U_i, \mathcal{O}|_{U_i})$$

as defined in Section 9.2.3.

10.13.3 Ratios of homogeneous polynomials of same degree dehomogenize to ratios of polynomials, and vice versa (up to benign factors)

We verify now that the homeomorphism $\varphi_i := \varphi_i|_{U_i} : U_i \to X_i$ of topological spaces is actually an isomorphism of k-spaces

$$\varphi_i : (U_i, \mathcal{O}|_{U_i}) \xrightarrow{\cong} (X_i, \mathcal{O}_{X_i}).$$

By Exercise 19, we must verify that φ_i preserves regularity in both directions, i.e., for each open $V \subset X_i$ and $f \in \mathrm{Func}(U, k)$ that with the notation

$$U := \varphi_i^{-1}(V), \quad f' := f \circ \varphi_i|_U,$$

one has

$$f \in \mathcal{O}_{X_i}(V) \iff f' \in \mathcal{O}(U).$$

We verify this as follows:

- In the forward direction, recall that f belongs to $\mathcal{O}_{X_i}(V)$ iff locally on V, f is a ratio of polynomials $g, h \in R_i$. In that case, it follows by homogenization (see especially Section 10.8.3) that f' is locally on U a ratio $x_i^{\deg(g^*)} g^* / (x_i^{\deg(h^*)} h^*)$ of homogeneous polynomials of the same degree, hence belongs to $\mathcal{O}(U)$.
- Conversely, suppose f' belongs to $\mathcal{O}(U)$. Then f' is locally on V a ratio g/h of homogeneous polynomials of the same degree. By dehomogenization as in Section 10.9.3, it follows that f is locally on U a ratio g_i/h_i of polynomials, hence belongs to $\mathcal{O}_{X_i}(V)$.

We conclude that any projective variety (equipped with the k-sheaf of regular functions) is a prevariety.

To summarize the above formal argument: given on U_i a local ratio of homogeneous polynomials of the same degree, we obtain (by setting $x_i := 1$) a local ratio of polynomials on X_i, and conversely, given a local ratio of polynomials on X_i, we get (after homogenizing and clearing denominators by suitable powers of x_i) a local ratio of homogeneous polynomials on U_i.

10.13.4 Coordinate ring of a basic affine patch

The isomorphism established in the previous section implies in particular that

$$\mathcal{O}(D(x_i)) = k\left[\frac{x_0}{x_i}, \dots \frac{x_n}{x_i}\right].$$

10.14 Projective varieties are varieties, i.e., are separated

We have seen that a projective variety X is naturally a prevariety. We now verify that this prevariety is actually separated. In other words, *any projective variety is a variety.* To see this, we must verify for each prevariety Z and pair of morphisms $f_1, f_2 : Z \to X$ that the subset $W := \mathrm{eq}(f_1, f_2)$ on which they coincide is closed in Z. Choose projective coordinates

$$[\alpha_0, \dots, \alpha_n] := f_1(z),$$

$$[\beta_0, \dots, \beta_n] := f_2(z)$$

for $f_1(z)$ and $f_2(z)$. (Note that although α_i and β_j depend upon z, we suppress this dependence for the sake of notational clarity.) For each pair of indices $k, l \in \{0, \ldots, n\}$, define the subset

$$Z_{kl} := \{z \in Z : \alpha_k \neq 0, \beta_\ell \neq 0\}$$

of Z. Since for each z at least one of the α_k and at least one of the β_ℓ is nonzero, we have $Z = \cup Z_{kl}$. Since $Z_{kl} = f_1^{-1}(D(x_k)) \cap f_2^{-1}(D(x_\ell))$, the sets Z_{kl} are open, hence give an open cover. Since being closed is a local notion, our task reduces to showing for each pair of indices k, l (which we fix for the remainder of this discussion) that

$$W_{kl} := W \cap Z_{kl}$$

is closed in Z_{kl}. Let $z \in Z_{kl}$. Since $\alpha_k \neq 0$, we have $z \in W_{kl}$ if and only if

$$\alpha_k \beta_i = \beta_k \alpha_i$$

for each i, which since $\beta_l \neq 0$ may be rearranged as

$$\frac{\beta_i}{\beta_l} = \frac{\beta_k}{\beta_l} \frac{\alpha_i}{\alpha_k}$$

and then expanded in terms of z as

$$x_{i/\ell}(f_2(z)) = x_{k/\ell}(f_2(z)) x_{i/k}(f_1(z))$$

where $x_{i/j} \in \mathcal{O}(D(x_j))$ denotes the regular function x_i/x_j, i.e.,

$$x_{i/j}([\gamma_0, \ldots, \gamma_n]) := \gamma_i/\gamma_j.$$

It follows that W_{kl} is the intersection over $i \in \{0, \ldots, n\}$ of the preimages of $\{0\}$ under the regular functions

$$x_{i/\ell} \circ f_2 - (x_{k/\ell} \circ f_2)(x_{i/k} \circ f_1)$$

on Z_{kl}, hence closed by the continuity of regular functions, as required.

Remark 78. The standard proofs of this result appearing in most references uses properties of the Segre embedding, which gives the categorical product of projective spaces. As an exercise, the reader is encouraged to complete the proof of Lemma 4.1 in Hartshorne by filling in the proof of the starred exercise that is referenced there. I have presented the above "brute force" proof for the sake of variety of exposition and to emphasize that this is a non-mysterious fact requiring no particular ingenuity to establish. We'll probably end up giving the standard proof a bit later in the course after we've properly discussed the Segre embedding. Slicker proofs are also possible. For instance, one can reduce quickly to showing that any pair of points in $\mathbb{P}^n$ are contained in some common affine open subset.

10.15 Homogeneous ideals

An ideal $\mathfrak{a} \subset k[x]$ is called *homogeneous* if it contains the homogeneous components of each of its elements, that is to say, if

$$\mathfrak{a} = \oplus_{d \in \mathbf{Z}_{\geq 0}} \mathfrak{a}_d, \qquad \text{where } \mathfrak{a}_d := \mathfrak{a} \cap k[x]_d.$$

Any ideal obtained as an intersection, product, sum, or radical of homogeneous ideals is also homogeneous.

A convenient criterion is that homogeneous ideals are precisely the ideals generated by sets consisting of homogeneous elements.

A projective variety may be alternatively defined as a set of the form $V_{\mathbb{P}^n}(\mathfrak{a})$ for some homogeneous ideal $\mathfrak{a} \subset k[x]_+$. Indeed, for any set S, the ideal $\mathfrak{a}$ generated by the homogeneous elements of S is a homogeneous ideal, and one has $V_{\mathbb{P}^n}(S) = V_{\mathbb{P}^n}(\mathfrak{a})$.

10.16 Quasi-projective varieties

A *quasi-projective variety* is an open subset of a projective variety, regarded as a k-space with the induced structure as in Section 9.2.3; thus the regular functions on an open subset are (once again) the k-valued functions that are locally ratios of homogeneous polynomials of the same degree. For example, any projective variety is a quasi-projective variety, and any open subset of a quasi-projective variety is a quasi-projective variety. By Example 71, any quasi-projective variety is a variety.

10.17 The projective vanishing ideal and the affine cone

10.17.1 Definition

We adopt here (the slightly nonstandard) definition that the *(projective) vanishing ideal* $I_{\mathbb{P}^n}(X)$ of a subset $X \subset \mathbb{P}^n$ is defined to be the ideal generated by those *positive-degree* homogeneous $f \in k[x]_+$ for which $f([\alpha]) = 0$ for all $[\alpha] \in X$, thus

$$I_{\mathbb{P}^n}(X) := (\{f \in k[x]_+ : \ f \text{ is homogeneous}, \ f([\alpha]) = 0 \text{ for all } [\alpha] \in X \ \}).$$

The projective vanishing ideal is thus a homogeneous ideal *contained in* $k[x]_+$, i.e.,

$$I_{\mathbb{P}^n}(X) = \oplus_{d \in \mathbf{Z}_{\geq 1}} I_{\mathbb{P}^n}(X)_d, \quad I_{\mathbb{P}^n}(X)_d = \{f \in k[x]_d : \ f([\alpha]) = 0 \text{ for all } [\alpha] \in X \ \}.$$

The definition given here differs from the standard definition in that we have required f to have positive degree, but the two definitions coincide unless X is the empty set, i.e.,

$$\text{for } X \neq \emptyset, \ I_{\mathbb{P}^n}(X) = (\{f \in k[x] : \ f \text{ is homogeneous}, \ f([\alpha]) = 0 \text{ for all } [\alpha] \in X \ \}).$$

Note also that the only homogeneous ideal of $k[x]$ not contained in $k[x]_+$ is the trivial ideal $(1) = k[x]$.

The associations $I_{\mathbb{P}^n}$ and $V_{\mathbb{P}^n}$ satisfy similar order-reversal, extremal, boolean, and ideal generation properties as in the affine case; also,

$$I_{\mathbb{P}^n}(V_{\mathbb{P}^n}(X)) = \mathrm{Zcl}(X).$$

Exercise 29. Verify that a homogeneous ideal $\mathfrak{a}$ is prime if and only if $fg \in \mathfrak{a}, f \notin \mathfrak{a} \implies g \in \mathfrak{a}$ for all *homogeneous* elements $f, g \in k[x]$. Deduce that a projective variety X is irreducible if and only if its projective vanishing ideal $I_{\mathbb{P}^n}(X)$ is prime.

10.17.2 Dilation-invariant subsets of $\mathbb{A}^{n+1}$ have homogeneous vanishing ideal

Let $X \subset \mathbb{A}^{n+1}$ be any subset which is *dilation-invariant* in the sense that

$$\alpha \in X, \lambda \in k^\times \implies \lambda\alpha \in X.$$

Then the vanishing ideal $I_{\mathbb{A}^{n+1}}(X)$ is *homogeneous.* To see this, suppose f vanishes on X and has the decomposition $f = \sum f_d$ into homogeneous components. Then for each $\alpha \in X$ and $\lambda \in k^\times$,

$$0 = f(\lambda\alpha) = \sum \lambda^d f_d(\alpha).$$

Because the field k is infinite, these relations taken over all λ imply that $f_d(\alpha) = 0$ for all d and all $\alpha \in X$. In summary,

$$f \in I_{\mathbb{A}^{n+1}}(X) \implies \text{each } f_d \in I_{\mathbb{A}^{n+1}}(X),$$

as required.

10.17.3 Comparison with the vanishing ideal of the affine cone

Given a subset $X \subset \mathbb{P}^n$ of projective space, we have two natural ways to obtain an ideal contained in $k[x]$:

- First, we may take the projective vanishing ideal

  $$I_{\mathbb{P}^n}(X),$$

  which we recall is the homogeneous ideal contained in $k[x]_+$ and generated by homogeneous polynomials of positive degree that vanish on X.

- Second, we may consider the affine cone $C_X \subset \mathbb{A}^{n+1}$, which we recall from Section 10.12 is given by $C_X := \{0\} \cup \pi^{-1}(X)$, and then take its (affine) vanishing ideal, which we denote for emphasis by

  $$I_{\mathbb{A}^{n+1}}(C_X).$$

It is useful to know that projective vanishing ideal of any subset of projective space is the same as the affine vanishing ideal of its affine cone:

$$I_{\mathbb{P}^n}(X) = I_{\mathbb{A}^{n+1}}(C_X).$$

(This would not quite be the case when $X = \emptyset$ with the standard definition of $I_{\mathbb{P}^n}$.) To verify this equality, set $\mathfrak{a} := I_{\mathbb{A}^{n+1}}(C_X)$. Since C_X contains 0, we know that $\mathfrak{a}$ is contained in $k[x]_+$. Since C_X is dilation-invariant, we know by Section 10.17.2 that $\mathfrak{a}$ is a homogeneous ideal. Therefore both the LHS and RHS of the equality to be verified are homogeneous ideals contained in $k[x]_+$, hence generated by homogeneous polynomials $f \in k[x]_d$ of positive degree $d > 0$. For any such f we have $f(0) = 0$, hence

$$f \in I_{\mathbb{A}^{n+1}}(C_X) \iff f(\alpha) = 0 \text{ for all } [\alpha] \in X$$

while

$$f \in I_{\mathbb{P}^n}(X) \iff f([\alpha]) = 0 \text{ for all } [\alpha] \in X.$$

These conditions coincide by the definition of the truth value of "$f([\alpha]) = 0$."

10.17.4 Every affine cone is the affine cone of a projective variety

By an *affine cone* $C \subset \mathbb{A}^{n+1}$ we shall mean an affine variety that contains $\{0\}$ and is dilation-invariant. By Section 10.17.2 and the assumption $0 \in C$, the vanishing ideal $I_{\mathbb{A}^{n+1}}(C)$ is homogeneous and contained in $k[x]_+$. Moreover, it is clear that for *any* homogeneous ideal $\mathfrak{a} \subset k[x]_+$, one has

$$C_{V_{\mathbf{P}^n}(\mathfrak{a})} = V_{\mathbb{A}^{n+1}}(\mathfrak{a}).$$

Therefore

$$C = C_X \quad \text{with } X := V_{\mathbb{P}^n}(I_{\mathbb{A}^{n+1}}(C)).$$

10.17.5 Homogeneous Nullstellensatz and varia

Suppose $\mathfrak{a} \subset k[x]_+$ is a positive-degree homogeneous ideal. Then

$$I_{\mathbb{P}^n}(V_{\mathbb{P}^n}(\mathfrak{a}) = I_{\mathbb{A}^{n+1}}(C_{V_{\mathbb{P}^n}(\mathfrak{a})}) = I_{\mathbb{A}^{n+1}}(V_{\mathbb{A}^{n+1}}(\mathfrak{a})) = r(\mathfrak{a}).$$

Moreover, the following are evidently equivalent:

- $V_{\mathbb{P}^n}(\mathfrak{a}) = \emptyset$
- $C_{V_{\mathbb{P}^n}(\mathfrak{a})} = \{0\}$
- $V_{\mathbb{A}^{n+1}}(\mathfrak{a}) = \{0\}$
- $r(\mathfrak{a}) = k[x]_+$.
- For each i, $\mathfrak{a}$ contains some power x_i^N of x_i.
- For each i, $\mathfrak{a}$ contains all sufficiently large powers of x_i.
- $\mathfrak{a}$ contains some graded piece $k[x]_d$ with $d > 0$.

10.17.6 Bijections

We obtain natural bijections between the following classes of objects:

- Affine cones $C \subset \mathbb{A}^{n+1}$.
- Radical homogeneous ideals $\mathfrak{a} \subset k[x]_+$.
- Projective varieties $X \subset \mathbb{P}^n$.

We get from one to the other via the maps

$$C \mapsto \mathfrak{a} := I_{\mathbb{A}^{n+1}}(C)$$

$$\mathfrak{a} \mapsto C := V_{\mathbb{A}^{n+1}}(\mathfrak{a})$$

$$\mathfrak{a} \mapsto X := V_{\mathbb{P}^n}(\mathfrak{a})$$

$$X \mapsto \mathfrak{a} := I_{\mathbb{P}^n}(X)$$

$$X \mapsto C := C_X := \{0\} \cup \pi^{-1}(X),$$

$$C \mapsto X := \pi(C - \{0\}) = V_{\mathbb{P}^n}(I_{\mathbb{A}^{n+1}}(C)).$$

That these maps are mutually inverse and form commutative triangles follows from what has been shown above.

It follows in particular that a projective variety is irreducible if and only if its affine cone is irreducible, as both conditions have already been shown equivalent to the corresponding ideal being prime.

10.18 The projective coordinate ring

The *projective coordinate ring* of a projective variety $X \subset \mathbb{P}^n$ is defined to be

$$S(X) := k[x]/I_{\mathbb{P}^n}(X),$$

regarded as a graded k-algebra:

$$S(X) = \oplus_{d \in \mathbf{Z}_{\geq 0}} S(X)_d.$$

It is generated by the degree one elements $\overline{x}_0, \dots, \overline{x}_n \in S(X)_1$ obtained as the images of $x_0, \dots, x_n \in k[x]_1$. In view of the identity $I_{\mathbb{P}^n}(X) = I_{\mathbb{A}^{n+1}}(C_X)$ of Section 10.17.3, we may alternatively define the projective coordinate ring to be the affine coordinate ring of the affine cone, i.e.,

$$S(X) = A(C_X).$$

In this optic, the grading is induced by the dilation action of $k^\times$ in the manner of Section 10.3.

10.19 Specifying morphisms to projective varieties in terms of homogeneous polynomials

Given a variety Y and a quasi-projective variety $X \subset \mathbb{P}^n$ and a function $f : Y \to X$, we say that f is *represented locally* as

$$f = [f_0, \dots, f_n]$$

with $f_0, \dots, f_n$ belonging to a specified class if for each $\beta \in Y$ there exists an open neighborhood $\beta \in V \subset Y$ and functions $f_0, \dots, f_n : V \to k$ of the prescribed class for which

- for each $\alpha \in V$ there is an index $i \in \{0..n\}$ so that $f_i(\alpha) \neq 0$, and
- $f(\alpha) = [f_0(\alpha), \dots, f_n(\alpha)]$ for all $\alpha \in V$, which identity we abbreviate as "$f = [f_0, \dots, f_n]$ on V."

With this terminology, we have the following equivalent characterizations of when f defines a morphism:

- f is locally of the form $f = [f_0, \dots, f_n]$ with regular functions $f_0, \dots, f_n$.
- For $X \subset \mathbb{P}^m$ a quasi-projective variety: f is locally of the form $f = [f_0, \dots, f_n]$ with homogeneous polynomials $f_0, \dots, f_n$ of the same degree.
- For $X \subset \mathbb{A}^m$ a quasi-affine variety: f is locally of the form $f = [f_0, \dots, f_n]$ with polynomials $f_0, \dots, f_n$.
- For X an irreducible variety: f is locally of the form $f = [f_0, \dots, f_n]$ with elements $f_0, \dots, f_n \in k(X)$ belonging to the function field of X (to be defined in some subsequent section; TODO: add a reference when that section is written).

The equivalence is left for now as an exercise.

11 The projective closure of an affine variety

11.1 Motivating question: how to describe how a variety looks "near infinity"?

We record here a detailed discussion of how one can attach to an affine variety $X \subset \mathbb{A}^n$ a projective variety $\overline{X} \subset \mathbb{P}^n$ and an inclusion $X \hookrightarrow \overline{X}$ whose complement is another projective variety $X_\infty \subset \mathbb{P}^{n-1}$ measuring the "asymptotic behavior" of X. This discussion serves the primary purpose of allowing us to exercise the concepts already introduced.

11.2 Affine space as a subspace of projective space

Via the isomorphism of varieties

$$\mathbb{A}^n \xrightarrow{\cong} \mathbb{P}^n_0 \xrightarrow{\cong} D(x_0) \subset \mathbb{P}^n$$

$$(\alpha_{1/0}, \ldots, \alpha_{n/0}) \mapsto (1, \alpha_{1/0}, \ldots, \alpha_{n/0}) \mapsto [1, \alpha_{1/0}, \ldots, \alpha_{n/0}]$$

obtained in Section 10.13, we may and shall identify $\mathbb{A}^n$ with the open subset $D(x_0)$ of $\mathbb{P}^n$. Its complement is the hyperplane

$$H_0 := V(x_0),$$

which is called the *hyperplane at* ∞ relative to the affine space $\mathbb{A}^n \cong D(x_0)$. We have the decomposition

$$\mathbb{P}^n = \mathbb{A}^n \sqcup H_0.$$

The natural map

$$H_0 \to \mathbb{P}^{n-1}$$

$$[0, \alpha_1, \ldots, \alpha_n] \mapsto [\alpha_1, \ldots, \alpha_n]$$

is an isomorphism (cf. Section 10.19).

In particular, any quasi-affine variety is (isomorphic to) a quasi-projective variety.

11.3 Definition of projective closure

Let us regard

$$\mathbb{A}^n = \operatorname{Specm} k[y_1, \ldots, y_n]$$

as embedded inside

$$\mathbb{P}^n \twoheadleftarrow \mathbb{A}^{n+1} - \{0\}, \quad \mathbb{A}^{n+1} = \operatorname{Specm} k[x_0, \ldots, x_n]$$

via the isomorphism $\mathbb{A}^n \cong D_{\mathbb{P}^n}(x_0)$ (as in Section 11.2) under which x_i/x_0 pulls back to y_i. Thus we identify

$$y_1 := x_1/x_0, \ldots, y_n := x_n/x_0$$

as regular functions on $D(x_0)$, and identify the points

$$\mathbb{A}^n \ni (\alpha_1, \ldots, \alpha_n) = [1, \alpha_1, \ldots, \alpha_n] \in \mathbb{P}^n.$$

Suppose given an affine variety $X \subset \mathbb{A}^n$; for example, one might suppose that $n = 2$ and that

$$X := V(y_1 y_2 - 1) \subset \mathbb{A}^2 \tag{13}$$

is a hyperbola with asymptotes the coordinate axes. We may regard X as a subset of $\mathbb{P}^n$ via the composite inclusion

$$X \subset \mathbb{A}^n \subset \mathbb{P}^n.$$

The closure of X in $\mathbb{P}^n$, denoted $\overline{X}$, is called the *projective closure* of X. It is a projective variety $\overline{X} \subset \mathbb{P}^n$. The difference

$$X_\infty := \overline{X} - X$$

is contained in the hyperplane at infinity $H_0 = V_{\mathbb{P}^n}(x_0)$, which is isomorphic to $\mathbb{P}^{n-1}$ via the map

$$H_0 \cong \mathbb{P}^{n-1}$$
$$[0, \alpha_1, \ldots, \alpha_n] \mapsto [\alpha_1, \ldots, \alpha_n].$$

We call X_∞ the *asymptotic part* of X (TODO: does this have a standard name?).

11.4 Example: the standard hyperbola and its asymptotes

Let X be the affine variety defined in (13), so that

$$X = \{(\gamma, \gamma^{-1}) = [1, \gamma, \gamma^{-1}] : \gamma \in k^\times\} \subset \mathbb{A}^2 \subset \mathbb{P}^2.$$

Define

$$g := x_1 x_2 - x_0^2.$$

The polynomial g is irreducible (by Eisenstein's criterion, for instance), so the ideal (g) is prime and the variety $V_{\mathbb{P}^2}(g)$ is irreducible. It is thus the closure of its relatively open subset

$$D(x_0) \cap V_{\mathbb{P}^2}(g) = \{[\alpha] = [1, \alpha_1, \alpha_2] : g(\alpha) = 0\} = \{[\alpha] = [1, \alpha_1, \alpha_2] : \alpha_1 \alpha_2 = 1\} = X.$$

Therefore $\overline{X} = V_{\mathbb{P}^2}(g)$. By the Nullstellensatz and the earlier observation that (g) is prime, it follows that

$$I_{\mathbb{P}^2}(\overline{X}) = I_{\mathbb{P}^2}(V_{\mathbb{P}^2}(g)) = (g).$$

The asymptotic part X_∞ is

$$X_\infty = \overline{X} - X = \{[\alpha] = [0, \alpha_1, \alpha_2] \in \mathbb{P}^2 : \alpha_1 \alpha_2 = 0\} = \{[0, 1, 0]\} \cup \{[0, 0, 1]\}$$

consisting of two "points at infinity" corresponding to the horizontal and vertical asymptotes of X inside $\mathbb{A}^2$.

11.5 Review of notation concerning homogenization and dehomogenization

Set

$$k[x] := k[x_0, x_1, \ldots, x_n],$$
$$k[y] := k[y_1, \ldots, y_n].$$

Recall the homogenization map $k[y] \ni f \mapsto f^* \in k[x]$ defined in Section 10.8, which in this notation is given by

$$f^*(x_0, \ldots, x_n) := x_0^{\deg(f)} f\left(\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}\right),$$

as well as the dehomogenization map $k[x] \ni f \mapsto f_{*0} \in k[y]$ defined in Section 10.9, which we abbreviate by $f_* := f_{*0}$ and recall in this notation as given by

$$f_*(y_1, \ldots, y_n) := f(1, y_1, \ldots, y_n).$$

For $f \in k[y]$, denote by

$$f^\top \in k[x_1, \ldots, x_n]$$

the homogeneous polynomial obtained by summing those monomials occurring in f of maximal degree and replacing each y_i with x_i. For example,

$$(y_1^2 - y_2 + 1 + 3y_3y_4)^\top = x_1^2 + 3x_3x_4.$$

11.6 Computation of vanishing ideal

The aim here is to "compute" $I_{\mathbb{P}^n}(\overline{X})$ in the relatively weak sense of finding any set of generators (not necessarily a finite set). The finer problem of determining a finite set of generators shall be addressed briefly in Section 11.8 and taken up subsequently. Regard the affine vanishing ideal $I_{\mathbb{A}^n}(X)$ as contained in $k[y]$. Observe right away that by definition of the Zariski closure,

$$I_{\mathbb{P}^n}(\overline{X}) = (\{ \text{ homogeneous } f \in k[x]_+ : f([\alpha]) = 0 \text{ for all } [\alpha] \in X\}),$$

hence that

$$I_{\mathbb{P}^n}(\overline{X}) = [f \mapsto f_*]^{-1}(I_{\mathbb{A}^n}(X)) = \{f \in k[x] : f_* \in I_{\mathbb{A}^n}(X)\}. \tag{14}$$

For

$$\alpha = (\alpha_1, \ldots, \alpha_n) = [1, \alpha_1, \ldots, \alpha_n] \in X$$

and $f \in I_{\mathbb{A}^n}(X)$, we calculate

$$f^*(\alpha) = 1^{\deg(f)} f\left(\frac{\alpha_1}{1}, \ldots, \frac{\alpha_n}{1}\right) = f(\alpha) = 0,$$

hence $f^*|_X = 0$. By definition of the Zariski closure, it follows that

$$f \in I_{\mathbb{A}^n}(X) \implies f^* \in I_{\mathbb{P}^n}(\overline{X}).$$

Conversely, for $h \in I_{\mathbb{P}^n}(\overline{X})$ we have $h|_X = 0$, hence $h_0|_X = 0$, i.e., $h_0 \in I_{\mathbb{A}^n}(X)$; since h is divisible by $(h_0)^*$, we deduce that h belongs to the ideal generated by $\{f^* : f \in I_{\mathbb{A}^n}(X)\}$.

We have thereby computed the projective vanishing ideal of the projective closure of an affine variety, firstly as the inverse image of the affine vanishing ideal under the dehomogenization map (see 14) and secondly as the ideal

$$I_{\mathbb{P}^n}(\overline{X}) = (\{f^* : f \in I_{\mathbb{A}^n}(X)\}) \subset k[x_0, \ldots, x_n]$$

generated by the homogenizations of elements of the original affine vanishing ideal.

It follows that the vanishing ideal of the asymptotic part

$$X_\infty = \overline{X} - X \subset H_0 \cong \mathbb{P}^{n-1},$$

being cut out inside $\overline{X}$ by the equation $x_0 = 0$, is defined by

$$X_\infty = V_{\mathbb{P}^{n-1}}(\{f^\top : f \in I_{\mathbb{A}^n}(X)\})$$

and hence has vanishing ideal

$$I_{\mathbb{P}^{n-1}}(X_\infty) = r((\{f^\top : f \in I_{\mathbb{A}^n}(X)\})).$$

11.7 Caution: the above descriptions are not particularly computationally useful

The "descriptions" of $I_{\mathbb{P}^n}(\overline{X})$ and $I_{\mathbb{P}^{n-1}}(X_\infty)$ obtained in Section 11.6 suffer the deficit that they do not furnish a *finite* set of generators for the respective projective vanishing ideals given some finite set of generators for the affine vanishing ideal $I_{\mathbb{A}^n}(X)$. We shall subsequently address this issue:

- In Section 11.10.5, we record an ad hoc computation of a finite set of defining equations for the projective closure of the twisted cubic curve $\{[\gamma, \gamma^2, \gamma^3] \in \mathbb{A}^3 : \gamma \in k\}$.
- In Section 11.11, we record a more sophisticated technique by which one may carry out such computations in general.

11.8 Caution: homogenizations of generators need not generate

It is not necessarily the case that for a collection of generators

$$I_{\mathbb{A}^n}(X) = (f_1, \dots, f_r)$$

of the vanishing ideal of the affine variety X that their homogenizations generate the projective closure $\overline{X}$, i.e., it is possible that

$$I_{\mathbb{P}^n}(\overline{X}) \neq (f_1^*, \dots, f_r^*).$$

An explicit example is recorded in Section 11.10.

11.9 The projective closure of an irreducible affine variety is an irreducible projective variety, and vice-versa

More generally, let X be a subset of a topological space T. Denote by $\overline{X}$ the closure of X inside T. Equip both X and $\overline{X}$ with the induced topology. Then X is irreducible if and only if $\overline{X}$ is irreducible; indeed, each of the following is evidently equivalent to the next:

- X is irreducible.
- Each nonempty open subset of X is dense.
- Each pair of proper closed subsets of X have proper union inside X.
- For each pair Z_1, Z_2 of closed subsets of T, one has

$$Z_1 \cup Z_2 \supset X \implies Z_1 \supset X \text{ or } Z_2 \supset X.$$

- For each pair Z_1, Z_2 of closed subsets of T, one has

$$Z_1 \cup Z_2 \supset \overline{X} \implies Z_1 \supset \overline{X} \text{ or } Z_2 \supset \overline{X}.$$

- $\overline{X}$ is irreducible.

In particular if $X \subset \mathbb{A}^n$ is an irreducible affine variety, then its projective closure $\overline{X} \subset \mathbb{P}^n$ is an irreducible projective variety. Conversely, if $Y \subset \mathbb{P}^n$ is an irreducible projective variety, then

$$Y_i := Y \cap D(x_i)$$

is an open subset of Y for each $i \in \{0..n\}$. It identifies with a closed subset of affine space via the isomorphism $\mathbb{A}^n \cong \mathbb{P}^n_i \cong D(x_i)$. Thus if $Y_i \neq \emptyset$, then

- Y_i is dense in Y, that is to say, $\overline{Y_i} = Y$, and
- Y_i is an irreducible affine variety.

11.10 Example: The twisted cubic

11.10.1 Definition

We follow here following Hartshorne, Exercise I.2.9. Consider the subset X of $\mathbb{A}^3$ defined by

$$X := \{(\gamma, \gamma^2, \gamma^3) \in \mathbb{A}^3 : \gamma \in k\}.$$

11.10.2 Computation of affine vanishing ideal

We record here two methods for computing $I_{\mathbb{A}^3}(X)$. The result obtained by either approach is that

$$I_{\mathbb{A}^3}(X) = \mathfrak{b}$$

where

$$\mathfrak{b} := (f_1, f_2) \subset k[y_1, y_2, y_3]$$

with

$$f_1 := y_2 - y_1^2,$$
$$f_2 := y_3 - y_1^3.$$

1. Direct approach Note first that any element of $\mathfrak{b}$ vanishes on X, since both f_1 and f_2 do. On the other hand:

   - For any $f \in k[y_1, y_2, y_3]$ that vanishes on X and is of the form $f(y_1, y_2, y_3) = h(y_1)$ for some $h \in k[y_1]$, we deduce from the identity $f(\gamma, \gamma^2, \gamma^3) = h(\gamma)$ that $h(\gamma) = 0$ for all $\gamma \in k$ and hence (by the infinitude of k) that $h = 0$.
   - For any $f \in k[y_1, y_2, y_3]$ that vanishes on X, we may subtract from it suitable multiples of f_1 or f_2 to obtain some f' vanishing on Y in which the variables y_2, y_3 do not occur. By the previous item, $f' = 0$, whence $f \in \mathfrak{b}$.

   It follows that $I_{\mathbb{A}^3}(X) = \mathfrak{b}$, as claimed.

2. Nullstellensatz approach Note now that

$$X = V_{\mathbb{A}^3}(\mathfrak{b}).$$

   Indeed, it is clear that $X \subset V_{\mathbb{A}^3}(\mathfrak{b})$; conversely, if $\beta = (\beta_1, \beta_2, \beta_3)$ belongs to $V_{\mathbb{A}^3}(\mathfrak{b})$, then from

$$f_1(\beta) = 0$$

   we obtain

$$\beta_2 = \beta_1^2$$

   and from

$$f_2(\beta) = 0$$

   we obtain

$$\beta_3 = \beta_1^3$$

   whence with $\gamma := \beta_1 \in k$ that

$$\beta = (\gamma, \gamma^2, \gamma^3) \in X.$$

   It follows by the Nullstellensatz that

$$I_{\mathbb{A}^3}(X) = I_{\mathbb{A}^3}(V_{\mathbb{A}^3}(\mathfrak{b})) = r(\mathfrak{b}),$$

   so to obtain the desired result it remains only to verify that

$$\mathfrak{b} = r(\mathfrak{b}),$$

   i.e., that $\mathfrak{b}$ is a *radical* ideal. For that, it suffices to show that $\mathfrak{b}$ is a *prime* ideal, i.e., that $k[y_1, y_2, y_3]/\mathfrak{b}$ is an integral domain. Indeed, we compute that

$$k[y_1, y_2, y_3]/\mathfrak{b} = \frac{k[y_1, y_2, y_3]}{(y_2 - y_1^2, y_3 - y_1^3)} \cong k[y_1],$$

   which is an integral domain, as required.

11.10.3 Explicit example in which homogenizations of generators need not generate

We have seen already that X is an affine variety with vanishing ideal

$$I_{\mathbb{A}^3}(X) = (f_1, f_2) \subset k[y_1, y_2, y_3]$$

with $f_1 := y_2 - y_1^2$, $f_2 := y_3 - y_1^3$. Let us now embed $\mathbb{A}^3$ in $\mathbb{P}^3$ as in Section 11, thereby identifying X with its image

$$X = \{[1, \gamma, \gamma^2, \gamma^3] \in \mathbb{P}^3 : \gamma \in k\}.$$

The homogenizations of f_1, f_2 are computed to be

$$f_1^* = x_0x_2 - x_1^2,$$

$$f_2^* = x_0^2x_3 - x_1^3.$$

Observe that the polynomials

$$g_1 := x_0x_3 - x_1x_2$$

and

$$g_2 := x_1x_3 - x_2^2$$

belong to $I_{\mathbb{P}^3}(\overline{X})$. Indeed, it suffices to verify that they vanish on X, which is true by inspection. However, neither g_1 nor g_2 belong to the ideal (f_1^*, f_2^*). This is best "verified in private;" one way is to note that under the quotient maps

$$\tau : k[x_0, x_1, x_2, x_3] \twoheadrightarrow k[x_0, x_1, x_2, x_3]/(x_0^2, x_1, x_2)$$

and

$$\tau' : k[x_0, x_1, x_2, x_3] \twoheadrightarrow k[x_0, x_1, x_2, x_3]/(x_0, x_1)$$

we have

$$\tau(g_1) = \tau(x_0x_3) \neq 0$$

but

$$\tau(f_1^*) = \tau(f_2^*) = 0$$

and

$$\tau'(g_2) = \tau'(x_2)^2 \neq 0$$

but

$$\tau'(f_1^*) = \tau'(f_2^*) = 0.$$

Thus

$$I_{\mathbb{P}^3}(\overline{X}) \neq (f_1^*, f_2^*) \text{ even though } I_{\mathbb{A}^3}(X) = (f_1, f_2).$$

11.10.4 Curious aside: computation of the projective variety cut out by the homogenization of the generators

The calculation

- a point $[\alpha] = [\alpha_0, \alpha_1, \alpha_2, \alpha_3] \in \mathbb{P}^3$ belongs to $V_{\mathbb{P}^3}(f_1^*, f_2^*)$ if and only if
  - $\alpha_0 \neq 0$ and thus (without loss of generality) $\alpha_0 = 1$ and thus $[\alpha] \in X$, or
  - $\alpha_0 = 0$ and thus $\alpha_1 = 0$

shows the more precise identity

$$V_{\mathbb{P}^3}(f_1^*, f_2^*) = Z \cup \overline{X}, \quad Z := \{[0, 0, \alpha_2, \alpha_3] : [\alpha_2, \alpha_3] \in \mathbb{P}^1\} = V_{\mathbb{P}^3}(x_0, x_1).$$

The observation

- the homogeneous polynomial g_2 defined in Section 11.10.3 vanishes on X, hence (by definition of Zariski closure) on $\overline{X}$, but not on Z

shows that $\overline{X}$ does not contain Z. Thus, in particular,

$$V_{\mathbb{P}^3}(f_1^*, f_2^*) \neq \overline{X},$$

which shows a stronger form of the counterexample recorded in the previous section.

11.10.5 Ad hoc computation of a finite set of defining equations for the projective closure

Our aim is now to verify that

$$I_{\mathbb{P}^3}(\overline{X}) = \mathfrak{a}$$

for the ideal $\mathfrak{a} \subset k[x] := k[x_0, x_1, x_2, x_3]$ defined by

$$\mathfrak{a} := (f_1^*, g_1, g_2) = (x_0 x_2 - x_1^2, x_0 x_3 - x_1 x_2, x_1 x_3 - x_2^2).$$

We record here a hands-on, ad hoc approach. Note right away that each generator f of $\mathfrak{a}$ satisfies $f_* \in I_{\mathbb{A}^3}(X)$, hence that

$$\mathfrak{a} \subset I_{\mathbb{P}^3}(\overline{X}).$$

The difficult part is to show the reverse inclusion.

Thus, take $f \in k[x_0, x_1, x_2, x_3]$ to be a homogeneous polynomial of some degree $d \geq 1$ which vanishes on X. We wish to show that f belongs to $\mathfrak{a}$. We rule out first the simpler case $d = 1$:

- If $d = 1$, then $f = \sum a_i x_i$ and $f(1, \gamma, \gamma^2, \gamma^3) = \sum a_i \gamma^i$, which vanishes for all $\gamma \in k$ only if each $a_i = 0$, whence $f = 0 \in \mathfrak{a}$.

Supposing henceforth that $d \geq 2$, let

$$x^i := x^{(i_0,i_1,i_2,i_3)} := x_0^{i_0} x_1^{i_1} x_2^{i_2} x_3^{i_3}.$$

be a monomial occurring in f. We wish to use the above generators of $\mathfrak{a}$ to "reduce" the monomial x^i modulo the ideal $\mathfrak{a}$ to some other monomial whose exponents are extremal in some lexicographic sense. Precisely:

- If $i_3 > 0$ and $i_1 > 0$, then

  $$x^i \equiv x^{(i_0,i_1-1,i_2+2,i_3-1)} \pmod{\mathfrak{a}},$$

  since the difference is manifestly divisible by $x_1 x_3 - x_2^2$.

- If $i_3 > 0$ and $i_0 > 0$, then

  $$x^i \equiv x^{(i_0-1,i_1+1,i_2+1,i_3-1)} \pmod{\mathfrak{a}},$$

  since the difference is manifestly divisible by $x_0 x_3 - x_1 x_2$.

- If $i_2 > 0$ and $i_0 > 0$, then

  $$x^i \equiv x^{(i_0-1,i_1+2,i_2-1,i_3)} \pmod{\mathfrak{a}},$$

  since the difference is manifestly divisible by $x_0 x_2 - x_1^2$.

By repeatedly applying the above operations, we may reduce any polynomial modulo $\mathfrak{a}$ to some linear combination of monomials x^i satisfying one of the following conditions:

1. $i_3 = 0$ and $i_2 = 0$, i.e., x^i involves only x_0, x_1.
2. $i_3 = 0$ and $i_2 > 0$ and $i_0 = 0$, i.e., x^i involves only x_1, x_2 and is divisible by x_2.
3. $i_3 > 0$ and $i_1 = i_0 = 0$, i.e., x^i involves only x_2, x_3 and is divisible by x_3.

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- Modulo $\mathfrak{a}$,

  $$x^{(100,200,3,4)} \equiv x^{(10,199,5,3)} \equiv \cdots \equiv x^{(100,196,11,0)} \equiv x^{(99,198,10,0)} \equiv \cdots \equiv x^{(89,218,0,0)}.$$

  We have reduced to case (1).

- Modulo $\mathfrak{a}$,

  $$x^{(10,20,3,4)} \equiv x^{(10,19,5,3)} \equiv \cdots \equiv x^{(10,16,11,0)} \equiv x^{(9,18,10,0)} \equiv \cdots \equiv x^{(0,36,1,0)}.$$

  We have reduced to case (2).

- Modulo $\mathfrak{a}$,

$$x^{(1,2,3,4)} \equiv x^{(1,1,5,3)} \equiv x^{(1,0,7,2)} \equiv x^{(0,1,8,1)} \equiv x^{(0,0,10,0)}.$$

  We have reduced again to case (2).

- Modulo $\mathfrak{a}$,

$$x^{(1,2,3,40)} \equiv x^{(1,1,5,39)} \equiv x^{(1,0,7,38)} \equiv x^{(0,1,8,37)} \equiv x^{(0,0,10,36)}.$$

  We have reduced to case (3).

General sums of degree d monomials of types (1), (2), (3) as above may be represented respectively as F, F', F' where

$$F(x_0, x_1, x_2, x_3) = G(x_0, x_1)$$

and

$$F'(x_0, x_1, x_2, x_3) = x_2 G'(x_1, x_2)$$

and

$$F''(x_0, x_1, x_2, x_3) = x_3 G''(x_2, x_3)$$

for some homogeneous polynomials G, G', G'' in the indicated variables of degrees $d, d-1, d-1$, respectively. By our reduction algorithm, f admits a representation

$$f \equiv F + F' + F'' \pmod{\mathfrak{a}}$$

for some such F, F', F''. The hypothesis $f(1, \gamma, \gamma^2, \gamma^3) = 0$ for all $\gamma \in k$ implies for the univariate polynomials $H, H', H'' \in k[t]$ defined by

$$H(t) := F(1, t, t^2, t^3) = G(1, t),$$

$$H'(t) := F'(1, t, t^2, t^3) = t^2 G'(t, t^2),$$

$$H''(t) := F''(1, t, t^2, t^3) = t^3 G''(t^2, t^3)$$

that

$$H + H' + H'' = 0.$$

Observe now that

- each monomial appearing in H has degree belonging to $\{0..d\}$,
- each monomial appearing in H' has degree belonging to $\{d + 1..2d\}$, and
- each monomial appearing in H'' has degree belonging to $\{2d + 1..3d\}$.

There is thus no monomial occurring in any two of H, H', H'' and hence no cancellation in the identity $H + H' + H'' = 0$. From this observation it follows that $H = H' = H'' = 0$ and so $f \equiv 0 \pmod{\mathfrak{a}}$, as required.

11.10.6 Remark: it's not always necessary to find a finite set of generators

We remark that for many purposes, it's not essential to find a finite set of generators for the vanishing ideal. For instance, in the present example one can verify quite easily that $I_{\mathbb{P}^3}(\overline{X})$ is the kernel of the graded homomorphism

$$k[x_0, x_1, x_2, x_3] \to k[w_0, w_1]$$

$$x_i \mapsto w_0^{3-i} w_1^i.$$

We shall not use this fact, and leave its verification as an exercise.

11.11 Using Groebner bases to compute a finite set of defining equations for the projective closure

We describe here a systematic approach generalizing and making less ad hoc the approach recorded above in Section 11.10.5. We loosely follow the treatment in Eisenbud's book "Commutative algebra with a view towards algebraic geometry."

11.11.1 Monomial orders

Let $S := k[x_0, \ldots, x_n]$. By a *monomial* in S we shall mean an element of the form

$$x_0^{a_0} \cdots x_n^{a_n}$$

for some $a_0, \ldots, a_n \in \mathbf{Z}_{\geq 0}$. By a *term* in S we shall mean a scalar multiple

$$c x_0^{a_0} \cdots x_n^{a_n}, \quad c \in k$$

of a monomial.

A *monomial order* on S is a total order "$<$" on the set of monomials $x^a = x_0^{a_0} \cdots x_n^{a_n}$ in S for which

- $x^a < x^b$ implies $x^{a+c} < x^{b+c}$ for all a, b, c, and
- $1 < x^a$ for all $a \neq 0$.

We extend any monomial order to a partial order on terms $cx^a, c'x^b$ with $c, c' \in k^\times$ by requiring that $cx^a < c'x^b$ if and only if $x^a < x^b$.

A good example to keep in mind is when $x^a > x^b$ is defined to happen if and only if either

- $\sum a_i > \sum b_i$ or
- $\sum a_i = \sum b_i$ and $a_i \neq b_i$ for some index i and for the smallest such index i, one has $a_i < b_i$.

When $n = 2$, this order is given explicitly by

$$1 < x_0 < x_1 < x_2 < x_0^2 < x_0x_1 < x_0x_2 < x_1^2 < x_1x_2 < x_2^2$$
$$< x_0^3 < x_0^2x_1 < x_0^2x_2 < x_0x_1^2 < x_0x_1x_2 < x_0x_2^2 < x_1^3 < \cdots,$$

and so on. In other words, monomials are ordered first by degree and then in "dictionary order" if one writes them down as $x_0^{a_0} \cdots x_n^{a_n}$. That's the only order we'll use. It's called *reverse lexicographic order*.[16]

Similar considerations here and in what follows apply with S replaced by $k[y_1, \ldots, y_n]$. Note that we did not really work with S as a graded ring in the above.

Here's a simple intuitive way to think about the reverse lexicographic order described above. Think

$$\text{“}x_i := t \log_i(t)\text{''}$$

where

$$\log_0(t) := 1, \quad \log_1(t) := \log(t), \quad \log_{i+1}(t) := \log(\log_i(t))$$

and t denotes a very large positive real, thought of as tending to $+\infty$. Then the above ordering on monomials corresponds to the usual ordering on the real numbers.

11.11.2 Initial terms

Recall that we have equipped S with a fixed monomial order.

Given a polynomial $f \in S$, its *initial term* is the term occurring in f whose underlying monomial is largest, with the convention $\text{in}(0) := 0$. For instance, if

$$f = 3x_0x_1 + 5x_1^2 + 20x_2$$

then for the reverse lexicographic ordering defined above,

$$\text{in}(f) = 5x_1^2.$$

This is usually denoted by underlying the initial term, i.e., by writing

$$f = 3x_0x_1 + \underline{5x_1^2} + 20x_2.$$

Note that with the reverse lexicographic ordering, $\text{in}(f) \in (x_0, \ldots, x_s)$ implies $f \in (x_0, \ldots, x_s)$ for homogeneous f. In fact, this property characterizes this ordering among all monomial orders that refine the partial order given by degree.

Note that taking the initial term is multiplicative, i.e.,

$$\text{in}(f_1f_2) = \text{in}(f_1)\,\text{in}(f_2) \text{ for all } f_1, f_2 \in S.$$

as follows from the basic properties of the monomial order. Moreover, for any monomial m,

$$\text{in}(m) = m,$$

[16] In some references this seems to be defined with the role of $\{0, 1, \ldots, n\}$ replaced by $\{n, n-1, \ldots, 0\}$.

hence for $f \in S$,

$$\operatorname{in}(mf) = m \operatorname{in}(f).$$

Taking the initial term is typically not additive in any simple sense, but it is "ultrametric" in the sense that

$$\operatorname{in}(f_1) > \operatorname{in}(f_2) \implies \operatorname{in}(f_1 + f_2) = \operatorname{in}(f_1).$$

We shall use these basic properties without explicit mention in what follows.

11.11.3 Initial ideals

Given an ideal $\mathfrak{a} \subset S$, the *initial ideal* of $\mathfrak{a}$, denoted $\operatorname{in}(\mathfrak{a})$, is the ideal generated by the initial terms of elements of $\mathfrak{a}$, i.e.,

$$\operatorname{in}(\mathfrak{a}) := (\{\operatorname{in}(f) : f \in \mathfrak{a}\}).$$

It is the prototypical example of a *monomial ideal*, i.e., an ideal generated by monomials. It is not yet clear how this ideal may be effectively computed. We shall return to that question later.

11.11.4 Standard monomials: those not appearing in the initial ideal

Let $\mathfrak{a} \subset S$ be an ideal. A monomial

$$x^a = x_0^{a_0} \cdots x_n^{a_n} \in k[x]$$

will be called *standard* (with respect to the ideal $\mathfrak{a}$) if it does not belong to the initial ideal $\operatorname{in}(\mathfrak{a})$. Thus, the standard monomials are precisely those that do not arise as initial terms of elements of $\mathfrak{a}$.

11.11.5 Macaulay's lemma: a basis for the quotient by an ideal is given by those monomial not appearing in the initial ideal

The set B of standard monomials is actually a (k-vector space) *basis* for the quotient $S/\mathfrak{a}$. To see this, we should verify that

1. B spans $S/\mathfrak{a}$, and that
2. B has linearly independent image in $S/\mathfrak{a}$.

To check that B spans, let $f \in S$ be given. Subtract from it all terms that are multiples of monomials B, that is to say, all terms that do not arise as the initial term of some element of $\mathfrak{a}$. The initial term of f is then also the initial term of some element of $\mathfrak{a}$. Subtracting off that multiple of $\mathfrak{a}$, we see that f is congruent modulo the span of B and $\mathfrak{a}$ to something with strictly smaller initial term. If we repeat this process sufficiently many times, we see that f belongs to the span of B and $\mathfrak{a}$. Here we have used that any monomial order is *artinian* in the sense that any descending chain is finite.

Conversely, to check independence, note that if f is any nonzero element of the span of B, i.e., any nonzero linear combination of monomials not belonging to $\operatorname{in}(\mathfrak{a})$, then in particular, the initial term of f does not belong to $\operatorname{in}(\mathfrak{a})$, hence $f \notin \mathfrak{a}$, as required.

11.11.6 Normal form with respect to an ideal (and a monomial order)

The main result of Section 11.11.5 may be restated as follows: Given an ideal $\mathfrak{a} \subset S$ and an element $f \in S$, there exists a unique linear combination f' of standard monomials for which

$$f \equiv f' \pmod{\mathfrak{a}}.$$

We refer to f' as the *normal form* of f with respect to $\mathfrak{a}$ (and the chosen monomial order). The proof in Section 11.11.5 gave an algorithm for computing normal forms.

For example, if $n = 0$, so that $S = k[x_0]$ and $\mathfrak{a}$ is the principal ideal

$$\mathfrak{a} = (x_0^d + \cdots) \subset k[x_0]$$

generated by some polynomial of degree $d \geq 0$, then the normal form of a typical polynomial

$$f = \sum_{i \geq 0} a_i x_0^i$$

is the polynomial

$$f = \sum_{i \in \{0..d-1\}} a_i x_0^i$$

obtained by throwing away any monomials divisible by x_0^d.

11.11.7 Testing equality under containment using initial ideals

Suppose given a pair of ideals $\mathfrak{a}, \mathfrak{b} \subset S$ with one contained in the other, say

$$\mathfrak{a} \subset \mathfrak{b}.$$

Suppose also that $\operatorname{in}(\mathfrak{a}) = \operatorname{in}(\mathfrak{b})$. Then in fact $\mathfrak{a} = \mathfrak{b}$. To see this, note (thanks to Section 11.11.5) that the set B of monomials not belonging to $\operatorname{in}(\mathfrak{a})$ (or equivalently, not belonging to $\operatorname{in}(\mathfrak{b})$) gives a common basis of both $S/\mathfrak{a}$ and $S/\mathfrak{b}$.

11.11.8 Initial terms of generators need not generate the initial ideal

Suppose given an ideal $\mathfrak{a} \subset S$ together with a finite set of generators $g_1, \ldots, g_t$, thus

$$\mathfrak{a} = (g_1, \ldots, g_t).$$

It is not in general the case that the initial elements of these generators generate the initial ideal, that is to say, it can happen that

$$\operatorname{in}(\mathfrak{a}) \neq (\operatorname{in}(g_1), \ldots, \operatorname{in}(g_t)).$$

For example, consider the ideal

$$\mathfrak{a} := (g_1, g_2)$$

given by the generators

$$g_1 := \underline{y_1^2} - y_2,$$

$$g_2 := \underline{y_1^3} - y_3$$

with their initial terms (with respect to the reverse lexicographic monomial order, as usual) underlined. Then $\mathfrak{a}$ contains the element

$$f := g_2 - y_1 g_1 = \underline{y_1 y_2} - y_3$$

whose initial term $y_1 y_2$ does not belong to the ideal generated by the initial terms y_1^2, y_1^3 of g_1, g_2, i.e.,

$$\mathrm{in}(f) \in \mathrm{in}(\mathfrak{a}), \quad \mathrm{in}(f) \notin (\mathrm{in}(g_1), \mathrm{in}(g_2)).$$

11.11.9 Definition of Groebner bases

A *Groebner basis* of an ideal $\mathfrak{a} \subset S$ is a generating set $\mathfrak{a} = (g_1, \ldots, g_t)$ for which

$$\mathrm{in}(\mathfrak{a}) = (\mathrm{in}(g_1), \ldots, \mathrm{in}(g_t)).$$

Concretely, this means the following: for each $f \in \mathfrak{a}$ there exists $i \in \{1..t\}$ and a term $cx^a = cx_0^{a_0} \cdots x_n^{a_n} \in k[x]$ so that

$$\mathrm{in}(f) = cx^a \, \mathrm{in}(g_i).$$

We postpone discussion of how to compute Groebner bases in practice until Section ???.

11.11.10 Initial term commutes with homogenization for the reverse lexicographic ordering

Set

$$k[y] := k[y_1, \ldots, y_n],$$

$$k[x] := k[x_0, x_1, \ldots, x_n],$$

and recall the homogenization map

$$k[y] \ni f \mapsto f^* \in k[x]$$

given by

$$f^*(x_0, \ldots, x_n) := x_0^{\deg(f)} f(x_1/x_0, \ldots, x_n/x_0).$$

Equip both $k[y]$ and $k[x]$ with the reverse lexicographic monomial order, and let $f \in k[y]$ be given. We can then do either of the following:

- take the initial term $\mathrm{in}(f) \in k[y]$ of f and then compute its homogenization $\mathrm{in}(f)^* \in k[x]$, or
- first compute the homogenization $f^* \in k[x]$ of f and then take its initial term $\mathrm{in}(f^*) \in k[x]$.

A useful property of the chosen monomial order is that we get the same result either way, i.e., that

$$\mathrm{in}(f^*) = \mathrm{in}(f)^* \quad \text{for all } f \in k[y].$$

For example, if

$$f := y_1^2 - y_2,$$

then

$$\begin{aligned} \mathrm{in}(f) &= y_1^2, \\ f^* &= x_1^2 - x_0 x_2, \\ \mathrm{in}(f^*) &= x_1^2, \\ \mathrm{in}(f)^* &= x_1^2. \end{aligned}$$

11.11.11 Recall: homogenizations of generators need not generate

Suppose given an affine variety $X \subset \mathbb{A}^n$ with affine vanishing ideal

$$\mathfrak{b} := I_{\mathbb{A}^n}(X) \subset k[y] := k[y_1, \ldots, y_n].$$

Denote by $\overline{X} \subset \mathbb{P}^n$ the projective closure of X and by

$$\mathfrak{a} := I_{\mathbb{P}^n}(X) \subset k[x] := k[x_0, \ldots, x_n]$$

its projective vanishing ideal (see Section ???). We have seen (see Section ???) that one has in general

$$\mathfrak{a} = (\{f^* : f \in \mathfrak{b}\}),$$

and the proof even gave the stronger statement

$$\{\text{ homogeneous elements of } \mathfrak{a}\} = \{x_0^d f^* : d \in \mathbf{Z}_{\geq 0}, f \in \mathfrak{b}\} \tag{15}$$

but also that there are examples in which

$$\mathfrak{a} \neq (g_1^*, \ldots, g_t^*) \text{ even though } \mathfrak{b} = (g_1, \ldots, g_t).$$

Of course, it is true in general that

$$(g_1^*, \ldots, g_t^*) \subset \mathfrak{a}.$$

The problem is that homogenizations of generators may fail to generate.

11.11.12 Homogenization of a Groebner basis is a Groebner basis and generates

On the other hand, the homogenizations of a Groebner basis

$$\mathfrak{b} = (g_1, \dots, g_t),$$

taken with respect to the usual reverse lexicographic monomial order, *do* generate the vanishing ideal of the projective closure: with

$$\mathfrak{g} := (g_1^*, \dots, g_t^*),$$

we have

$$\mathfrak{g} = \mathfrak{a}.$$

To see this, it suffices by the containment $\mathfrak{g} \subset \mathfrak{a}$ observed in Section 11.11.11 and the result of Section 11.11.7 to verify that

$$\operatorname{in}(\mathfrak{g}) = \operatorname{in}(\mathfrak{a}).$$

To that end, it suffices to show for each homogeneoues $f \in \mathfrak{a}$ the stronger statement that there exists $i \in \{1..t\}$ and a term $cx^a = cx_0^{a_0} \cdots x_n^{a_n} \in k[x]$ so that

$$\operatorname{in}(f) = cx^a \operatorname{in}(g_i^*).$$

(This statement implies moreover that the generators $g_1^*, \dots, g_t^*$ give a *Groebner basis* of $\mathfrak{a}$.) We know that

$$f = x_0^d (f_*)^*,$$

where $d \in \mathbf{Z}_{\geq 0}$ is the order to which x_0 divides f, and also that f_* belongs to $\mathfrak{b}$. Since $\mathfrak{b} = (g_1, \dots, g_t)$ is a Groebner basis, there exists $i \in \{1..t\}$ and a term $cy^a = cy_1^{a_1} \cdots y_n^{a_n} \in k[y]$ so that

$$\operatorname{in}(f_*) = cy^a \operatorname{in}(g_i)$$

It follows from the remark of Section 11.11.2 and the result of Section 11.11.10 that

$$\operatorname{in}(f) = \operatorname{in}(x_0^d (f_*)^*) = x_0^d \operatorname{in}(f_*)^* = x_0^d (cy^a \operatorname{in}(g_i))^* = cx_0^d x_1^{a_1} \cdots x_n^{a_n} \operatorname{in}(g_i^*),$$

as required.

11.11.13 Division algorithm

Observe that if one is given a Groebner basis $\mathfrak{a} = (g_1, \dots, g_t)$, then one may use it to compute the normal forms (see Section 11.11.6) of arbitrary elements $f \in S$ by means of the following algorithm:

- If $f = 0$, we are done.

- If $f \neq 0$, then the initial term $\mathrm{in}(f)$ of f is either a standard monomial, in which case we subtract if off, or of the form $m\,\mathrm{in}(g_i) = \mathrm{in}(mg_i)$ for some term $m := cx_0^{a_0} \cdots x_n^{a_n}$; in the latter case, the difference $f - mg_i$ has strictly smaller initial term than f, so we may compute its normal form recursively.

The same algorithm shows more generally the following: given any elements $f, g_1, \dots, g_t \in S$, one may write (non-uniquely)

$$f = \sum f_i g_i + f'$$

where

- f' has no monomial contained in $(\mathrm{in}(g_1), \dots, \mathrm{in}(g_t))$, and
- each $\mathrm{in}(f_i g_i) \leq \mathrm{in}(f)$.

An element $f' \in S$ obtained in this way is called a *remainder* of f with respect to $g_1, \dots, g_t$ (and the chosen monomial order). Some examples will arise in the following section.

Note that $(g_1, \dots, g_t) = \mathfrak{a}$ is a Groebner basis if and only if for each $f \in \mathfrak{a}$, any remainder of f with respect to $g_1, \dots, g_t$ is zero. In the forward direction, suppose that the indicated basis is a Groebner basis. Then for each $f \in \mathfrak{a}$, any remainder f' belongs to $\mathfrak{a}$ and has no monomial contained in $(\mathrm{in}(g_1), \dots, \mathrm{in}(g_t))$; the latter ideal coincides with $\mathrm{in}(\mathfrak{a})$ by hypothesis, whence $f' = 0$. Conversely, suppose each such remainder vanishes. We wish then to verify that the indicated basis is a Groebner basis. A basis for $\mathrm{in}(\mathfrak{a})$ is given by the initial terms $\mathrm{in}(f)$ of the elements $f \in \mathfrak{a}$, so it suffices to show that each such initial term belongs to $(\mathrm{in}(g_1), \dots, \mathrm{in}(g_t))$. The latter assertion is evident when $f = 0$, so consider henceforth the case $f \neq 0$, hence $\mathrm{in}(f) \neq 0$. By the assumption about the vanishing of remainders, we may write $f = \sum f_i g_i$ where each $\mathrm{in}(f_i g_i) \leq \mathrm{in}(f)$. It follows that some $\mathrm{in}(f_i g_i)$ is a scalar multiple of $\mathrm{in}(f)$. Since $\mathrm{in}(f_i g_i) = \mathrm{in}(f_i)\,\mathrm{in}(g_i)$ belongs to $(\mathrm{in}(g_1), \dots, \mathrm{in}(g_t))$, we obtain the required conclusion.

11.11.14 Buchberger's criterion

There is a simple criterion to test whether a given generating set $\{g_1, \dots, g_t\}$ for an ideal

$$\mathfrak{a} = (g_1, \dots, g_t)$$

is a Groebner basis. For each $i, j \in \{1..t\}$, define g_{ij} to be (intuitively) the "smallest combination of g_i and g_j that kills off the leading terms," formalized as follows: set

$$m_{ij} := \frac{\mathrm{in}(g_i)}{\gcd(\mathrm{in}(g_i), \mathrm{in}(g_j))}$$

and

$$g_{ij} := m_{ji} g_i - m_{ij} g_j \in \mathfrak{a}.$$

Note that $g_{ji} = -g_{ij}$. Since the leading terms in the definition of g_{ij} cancel (by construction), we have

$$\mathrm{in}(g_{ij}) < \mathrm{in}(m_{ji}g_i).$$

Note also by comparing leading terms that whenever two monomials a_i, a_j satisfy

$$a_i\,\mathrm{in}(g_j) = a_j\,\mathrm{in}(g_i),$$

one has

$$a_i g_j - a_j g_i = c g_{ij}$$

for some monomial c. For example, if

$$g_1 := \underline{y_1^2} - y_2$$

and

$$g_2 := \underline{y_1^3} - y_3$$

then $m_{12} = 1, m_{21} = y_1 n$ and

$$g_{12} = m_{21}g_1 - m_{12}g_2 = y_1 g_1 - g_2 = y_3 - \underline{y_1 y_2}.$$

whose leading term indeed satisfies

$$y_1 y_2 < y_1^3.$$

Denote by

$$h_{ij} \in \mathfrak{a}$$

any remainder for g_{ij} with respect to $g_1, \ldots, g_t$; this means that h_{ij} contains no monomial in $(\mathrm{in}(g_1), \ldots, \mathrm{in}(g_t))$ and that we may write

$$g_{ij} = \sum_k f_k^{ij} g_k + h_{ij}$$

for some $f_k^{ij} \in S$ satisfying

$$\mathrm{in}(f_k^{ij} g_k) \leq \mathrm{in}(g_{ij}) < \mathrm{in}(m_{ji}g_i).$$

For instance, continuing the above example, we have with respect to the collection $\{g_1, g_2\}$ that

$$h_{12} = g_{12} = y_3 - \underline{y_1 y_2}.$$

If we add to our collection the third element

$$g_3 := \underline{y_1 y_2} - y_3$$

and now define h_{ij} with respect to $\{g_1, g_2, g_3\}$, then:

$$g_{12} = y_1 g_1 - g_2 = -g_3,$$

whence

$$h_{12} = 0;$$

$$g_{13} = y_2 g_1 - y_1 g_3 = y_1 y_3 - \underline{y_2^2}$$

with $\underline{y_2^2} \notin (\mathrm{in}(g_1), \mathrm{in}(g_2), \mathrm{in}(g_3)) = (y_1^2, y_1^3, y_1 y_2) = (y_1^2, y_1 y_2)$, whence

$$h_{13} = y_1 y_3 - \underline{y_2^2};$$

and

$$g_{23} = y_2 g_2 - y_1^2 g_3 = y_1^2 y_3 - y_2 y_3 = y_3 g_1,$$

whence

$$h_{23} = 0.$$

On the other hand, if we add a fourth element

$$g_4 := \underline{y_2^2} - y_1 y_3$$

and recompute the remainders h_{ij} now with respect to the larger set $\{g_1, g_2, g_3, g_4\}$, then we find that each $h_{ij} = 0$. This calculation shall be revisited below in Section ???.

Theorem 80. *The following are equivalent:*

- $(g_1, \ldots, g_t)$ *is a Groebner basis for* $\mathfrak{a}$.
- *Each* $h_{ij} = 0$.

This criterion gives a simple way to determine whether any collection of polynomials form a Groebner basis for the ideal they generate. For instance, with the example developed above, we have

$$\mathfrak{a} = (g_1, g_2) = (g_1, g_2, g_3) = (g_1, g_2, g_3, g_4).$$

The calculations recorded above imply by this criterion that

- $\{g_1, g_2\}$ is not a Groebner basis (because $h_{12} \neq 0$),
- $\{g_1, g_2, g_3\}$ is not a Groebner basis (because $h_{13} \neq 0$), and
- $\{g_1, g_2, g_3, g_4\}$ is a Groebner basis (because each $h_{ij} = 0$).

Moreover, we see that we can always extend a given basis to a Groebner basis by repeatedly adjoining the nonzero remainders h_{ij} that arise in such calculations. We thereby obtain the promised algorithm for computing Groebner bases.

11.11.15 Proof of Buchberger's criterion

The forward implication is immediate (as in the final remark of Section 11.11.13) from the assumptions that $h_{ij} \in \mathfrak{a}$ and that h_{ij} contains no monomial in $(\mathrm{in}(g_1), \dots, \mathrm{in}(g_t))$, the latter of which equals $\mathrm{in}(\mathfrak{a})$ by the hypothesis that $(g_1, \dots, g_t)$ is a Groebner basis.

Let us now explain the reverse implication. Supposing each $h_{ij} = 0$, the above representation simplifies to

$$g_{ij} = \sum_k f_k^{ij} g_k.$$

To show that $(g_1, \dots, g_t)$ is a Groebner basis, we must verify for each given $a \in \mathfrak{a}$ that its initial term $\mathrm{in}(a)$ belongs to $(\mathrm{in}(g_1), \dots, \mathrm{in}(g_t))$. We'll verify this by giving an algorithm, which we have optimized here for readability rather than practicality:

1. Take as input a representation $f = \sum a_k g_k$ for some $a_k \in S$. The aim is to obtain a representation for $\mathrm{in}(f)$ as an S-linear combination of the $\mathrm{in}(g_k)$.

2. By linearity, it suffices to consider the case that each a_k is a monomial and that the initial terms $\mathrm{in}(a_k g_k)$ of the nonzero summands of $\sum a_k g_k$ are all scalar multiples of the same monomial, call it m, say

   $$\mathrm{in}(a_k g_k) = \mu_k m$$

   for some scalar μ_k. In what follows we implicitly restrict to indices for which $\mu_k \neq 0$.

3. If there is no cancellation among the initial terms of the summands $a_k g_k$, that is to say, if

   $$\sum \mathrm{in}(a_k g_k) = (\sum \mu_k) m \neq 0,$$

   then the ultrametric property of taking initial terms implies that we must have

   $$\mathrm{in}(f) = \sum \mathrm{in}(a_k g_k) = \sum \mathrm{in}(a_k) \, \mathrm{in}(g_k),$$

   which gives the desired representation of $\mathrm{in}(f)$ in terms of the $\mathrm{in}(g_k)$.

4. It remains to consider the case that $\sum \mu_k = 0$. It suffices in that case by recursion to find a representation

   $$\sum a_k g_k = \sum b_k g_k$$

   where the $b_k \in S$ have the property that

   $$\mathrm{in}(b_k g_k) < m.$$

To obtain such a representation, first choose scalars ν_{ij} so that for any collection of elements $\varepsilon_1, \ldots, \varepsilon_t$ of some k-vector space, one has

$$\sum_k \mu_k \varepsilon_k = \sum_{i,j} \nu_{ij}(\varepsilon_i - \varepsilon_j).$$

For instance, it suffices to take $\nu_{i,j} := 0$ unless $(i,j) = (i, i+1)$ for some $i \in \{1..t-1\}$, in which case $\nu_{i,i+1} := \mu_1 + \mu_2 + \cdots + \mu_i$. Then

$$\sum a_k g_k = \sum \mu_k \frac{a_k g_k}{\mu_k} = \sum_{i,j} \nu_{ij} d_{ij}, \quad d_{ij} := \frac{a_i g_i}{\mu_i} - \frac{a_j g_j}{\mu_j}.$$

From the relation $\operatorname{in}(d_{ij}) = 0$ it follows by comparing leading terms that there exists a term $c_{ij} \in S$ for which

$$d_{ij} = c_{ij} g_{ij}.$$

Since the leading terms of the fractions in the definition of d_{ij} cancel and each $\operatorname{in}(a_k g_k) = m$, we have

$$\operatorname{in}(d_{ij}) < m.$$

On the other hand, in the decomposition

$$g_{ij} = \sum_k f_{ij}^k g_k,$$

we know that each summand satisfies

$$\operatorname{in}(f_{ij}^k g_k) \leq \operatorname{in}(g_{ij}).$$

It follows that

$$\operatorname{in}(c_{ij} f_{ij}^k g_k) \leq \operatorname{in}(c_{ij} g_{ij}) = \operatorname{in}(d_{ij}) < \operatorname{in}(m).$$

With the definition

$$b_k := \sum_{i,j} \nu_{ij} c_{ij} f_{ij}^k$$

we deduce that

$$\sum a_k g_k = \sum b_k g_k$$

and

$$\operatorname{in}(b_k g_k) < m,$$

as required.

11.11.16 The twisted cubic revisited

Aimed with this new machinery, let us see how much simpler it becomes to compute the projective closure of the twisted cubic curve $X := \{[\gamma, \gamma^2, \gamma^3] : \gamma \in k\}$. (Many of the calculations here are essentially repeats of those in the examples of Section 11.11.14, but we record them anyway.) We compute as before that $\mathfrak{a} := V_{\mathbb{A}^3}(X)$ is given by

$$\mathfrak{a} := (y_2 - y_1^2, y_3 - y_1^3) = (g_1, g_2, g_3),$$

where

$$g_1 := \underline{y_1^2} - y_2,$$
$$g_2 := \underline{y_1 y_2} - y_3,$$
$$g_3 := \underline{y_2^2} - y_1 y_3.$$

Using the division algorithm, we compute

$$g_{12} = y_2 g_1 - y_1 g_2 = y_1 y_3 - \underline{y_2^2} = -g_3,$$

whence

$$h_{12} = 0;$$

$$g_{13} = y_2^2 g_1 - y_1^2 g_3 = \underline{y_1^3 y_3} - y_2^3 = y_1 y_3 g_1 + y_1 y_2 y_3 - \underline{y_2^3} = y_1 y_3 g_1 - y_2 g_2,$$

whence

$$h_{13} = 0;$$

and

$$g_{23} = y_2 g_2 - y_1 g_3 = \underline{y_1^2 y_3} - y_2 y_3 = y_3 g_1,$$

whence

$$h_{23} = 0.$$

We conclude that $\{g_1, g_2, g_3\}$ is a Groebner basis for $\mathfrak{a}$ and hence that

$$I_{\mathbb{P}^3}(\overline{X}) = (g_1^*, g_2^*, g_3^*),$$

thereby recovering (much more quickly) the result derived earlier in Section 11.10.5 by ad hoc methods.

Exercise 30. Compute a Groebner basis for $\mathfrak{a} := (\underline{y_2} - y_1^3, y_3 - y_1^4)$ and use this to find defining equations for the projective closure $\overline{X}$ of $X := V_{\mathbb{A}^3}(\mathfrak{a})$ inside $\mathbb{P}^3$ and also its asymptotic part $X_\infty := \overline{X} - X \subset \mathbb{P}^2$.

11.11.17 Extension to free modules

One can extend all of the above discussion to free modules over S (in place of S itself); see Eisenbud's book.

11.11.18 Use a computer

For any remotely serious computations it is recommended that one use a computer.

12 Some topological review

We briefly review some basic topological notions and collect together some facts that will be convenient in what follows.

12.1 Density

12.1.1 Definitions of being dense

Let T be a topological space. Recall that a subset W of T is *dense* if it satisfies any of the following equivalent conditions:

- Each nonempty open $U \subset T$ intersects W.
- Each proper closed $Z \subset T$ fails to contain W.
- The only closed $Z \subset T$ containing W is $Z = T$.
- The closure of W is T.
- The only open $U \subset T$ contained in the complement of W is $U = \emptyset$.
- The interior of the complement of W is the empty set.

12.1.2 Density is preserved upon passing to open subsets

Let X be a subset of a topological space T, equipped with the induced topology, and let $D \subset T$ be a dense subset. It need not in general be the case that $D \cap X$ is dense in X. For example, take

$$T := \mathbb{R}, \quad X := \mathbb{R} - \mathbb{Q}, \quad D := \mathbb{Q}.$$

Then $D \cap X = \emptyset$; since $X \neq \emptyset$, it follows that $D \cap X$ is not dense in X.

However, if X is *open* in T, then $D \cap X$ is necessarily dense in X whenever D is dense in T. This follows immediately from the fact that the open subsets $U \subset X$ are just the open subsets of T that are contained in U.

12.1.3 Density is local

Suppose we have a topological space T and an open cover $T = \cup U_\alpha$ and a subset $D \subset T$ with the property that $D \cap U_\alpha$ is dense in U_α for each α. Then D is necessarily dense in T. To see this, let $W \subset T$ be open. Choose α for which $W \cap U_\alpha \neq \emptyset$. Then $W \cap U_\alpha$ is nonempty and open, and so D intersects it, whence a fortiori D intersects W, as required.

12.1.4 Density is transitive

Let $f : X \to Y$ be a continuous map of topological spaces with dense image, i.e., for which $f(X)$ is dense in Y. Let $D \subset X$ be dense. Then $f(D)$ is also dense in Y. Indeed, any nonempty open $W \subset Y$ has the property that $f(X) \cap W \neq \emptyset$, whence $f^{-1}(W) \neq \emptyset$. Since f is continuous, the preimage $f^{-1}(W)$ is nonempty open. Since D is dense, $f^{-1}(W) \cap D \neq \emptyset$.

In particular, for a topological space T and subsets $A \subset B \subset T$ equipped with the induced topology for which A is dense in B and B is dense in T, we deduce that A is dense in T by applying the preceeding considerations to the (continuous) inclusion map $B \hookrightarrow T$.

12.1.5 Finite intersections of dense open subsets are dense and open

It is not necessarily the case that a pair of dense subsets W_1, W_2 of T have dense intersection $W_1 \cap W_2$; it can even happen that $W_1 \cap W_2 = \emptyset$, as when for instance

$$T := \mathbb{R}, \quad W_1 := \mathbb{Q}, \quad W_2 := \sqrt{2} + \mathbb{Q}.$$

On the other hand, if W_1, W_2 are dense *open* subsets of T, then $W_1 \cap W_2 \subset T$ is dense and open. Indeed, it follows from Section 12.1.2 that $W_1 \cap W_2$ is dense in W_1 and then from Section 12.1.4 that $W_1 \cap W_2$ is dense in T. By iterating this argument, we see that any finite collection $U_1, \dots, U_n$ of dense open subsets of T have dense open intersection $U_1 \cap \dots \cap U_n \subset T$.

12.2 Irreducibility

12.2.1 Definitions

Recall that a topological space T is *irreducible* if equivalently

- any nonempty open $U \subset T$ is dense,
- any pair of nonempty opens $U, V \subset T$ intersect,
- T cannot be expressed as the union of two proper closed subsets, that is to say, any pair of closed proper subsets $W, Z \subsetneq T$ have proper union $W \cup Z \subsetneq T$, or
- for any decomposition $T = W_1 \cup W_2$ with $W_1, W_2 \subset T$ closed, one has either $W_1 = T$ or $W_2 = T$.

Exercise 31. Rewrite the proofs of the following sections using a different characterization of irreducibility than the one I used (e.g., rewrite the proof that I wrote by taking complements).

12.2.2 A set is irreducible iff its closure is irreducible

We saw in Section 11.9 that if $X \subset T$, then X is irreducible if and only if its closure $\overline{X}$ is irreducible. In particular, any nonempty open subset of an irreducible space is itself irreducible.

12.2.3 Being irreducible is sort of a local condition

Suppose given a topological space T with an open cover $T = \cup U_\alpha$ and a subset $X \subset T$ with the property that $X \cap U_\alpha$ is irreducible (with respect to the induced topology, as usual) for each α. Then it need not be the case that X is itself irreducible; consider for instance the case that X is a disjoint union of two points. However, if we assume also that each pairwise intersection is nonempty, i.e., that

$$U_{\alpha\beta} := U_\alpha \cap U_\beta \neq \emptyset \text{ for all } \alpha, \beta,$$

then X is irreducible. Indeed, let V, V' be nonempty open subsets of X. We must show that they have nonempty intersection. Choose α, β so that $W := U_\alpha \cap V$ and $W' := U_\beta \cap V'$ are nonempty. Since U_α, U_β are irreducible, we know that W is dense in U_α and W' is dense in U_β. By Section 12.1.2, it follows that $U_{\alpha\beta} \cap W$ and $U_{\alpha\beta} \cap W'$ are both dense in $U_{\alpha\beta}$. In particular, they have nonempty intersection. This implies $V \cap V' \neq \emptyset$, as required.

12.2.4 The image of an irreducible space is irreducible

Let $f : Y \to X$ be a continuous map between topological spaces, and let $Z \subset Y$ be irreducible. Then $f(Z) \subset X$ is also irreducible. Indeed, note first (by definition of the induced topology on $f(Z)$) that any pair of nonempty open subsets of $f(Z)$ are of the form $f(Z) \cap U_1, f(Z) \cap U_2$ for some open $U_1, U_2 \subset X$ which intersect $f(Z)$. Since f is continuous, it follows that $f^{-1}(U_1), f^{-1}(U_2)$ are open subsets of X which intersect Z. Suppose for the sake of contradiction that $f(Z) \cap U_1, f(Z) \cap U_2$ do not intersect. By definition of the induced topology on Z, we deduce that

$$Z \cap f^{-1}(U_1), \quad Z \cap f^{-1}(U_2)$$

are nonempty non-intersecting open subsets of Z. Since Z is irreducible, we derive the required contradiction.

12.2.5 Closure of image of closed irreducible is closed irreducible

A consequence of the discussion of Sections 12.2.2 and 12.2.4 is that if $f : Y \to X$ is a continuous map of topological spaces and $V \subset Y$ is a closed irreducible subset, then $\overline{f(V)} \subset X$ is also a closed irreducible subset.

13 Uniqueness of limits

13.1 Review of the basic principle

Let us spare a few words to elaborate on the meaning of the following immediate consequence of the definition of "separated" (see Section 9.5):

Let Y be a prevariety, let $U \subset Y$ be a dense open subset, let X be a variety, and let $f_1, f_2 : Y \to X$ be morphisms with the property that $f_1|_U = f_2|_U$. Then $f_1 = f_2$.

What this result formalizes is the notion that "limits are unique," that is to say, given a morphism $f : U \to X$ where U is a dense open subset of Y and X is separated, there is at most one extension of f to a morphism $Y \to X$. Such an extension may or may not exist, but if one does exist, then it is uniquely determined by f.

In particular, given an irreducible prevariety Y and a variety X and a nonempty open $U \subset Y$ and two morphisms $f, g : U \to X$ for which $f|_U = g|_U$, we have $f = g$. (The point here is just that any nonempty open subset of an irreducible variety is dense.)

13.2 Example: the line through the origin and a nonzero point in the plane

To give a simple example, set

$$Y := \mathbb{A}^2,$$

$$X := \mathbb{P}^1,$$

$$U := \mathbb{A}^2 - \{O\}, \quad O := (0, 0),$$

and consider the morphism

$$\pi : U \to X$$

$$(\alpha_1, \alpha_2) \mapsto [\alpha_1, \alpha_2].$$

Thus π maps a nonzero point P in the affine plane to the line contain P and the origin O. It is intuitively plausible that one cannot extend π to a morphism $Y \to X$ defined also at the origin; the problem is that one can approach the origin from several different directions. To formalize this, suppose otherwise that there does exist a morphism

$$f : Y \to X$$

for which $f|_U = \pi$. Let $\beta \in k$ be given, and consider the morphism

$$j_\beta : \mathbb{A}^1 \to \mathbb{A}^2$$

$$\gamma \mapsto [\gamma, \beta\gamma]$$

which parametrizes a line with slope β. Note that $j_\beta(\gamma) \in U$ for all $\gamma \in \mathbb{A}^1 - \{0\}$. Then the composition

$$f \circ j_\beta : \mathbb{A}^1 \to \mathbb{P}^1$$

has the property that for all $\gamma \in \mathbb{A}^1 - \{0\}$,

$$(f \circ j_\beta)(\gamma) = f(j_\beta(\gamma)) = \pi(j_\beta(\gamma)) = \pi([\gamma, \beta\gamma]) = [\gamma, \beta\gamma] = [1, \beta].$$

Thus $f \circ j_\beta$ coincides on the dense open subset $\mathbb{A}^1 - \{0\}$ of $\mathbb{A}^1$ with the constant morphism

$$p_\beta : \mathbb{A}^1 \to \mathbb{P}^1$$

$$\gamma \mapsto [1, \beta].$$

Since $\mathbb{P}^1$ is separated, it follows that the $f \circ j_\beta = p_\beta$ on all of $\mathbb{A}^1$. Therefore

$$f(O) = (f \circ j_\beta)(0) = p_\beta(0) = [1, \beta].$$

But since $[1, \beta]$ takes different values as β varies through the field k, we derive the required contradiction.

13.3 Example: tending off to infinity

Now take

$$Y := \mathbb{A}^2,$$
$$U := D_X(x_1),$$
$$X := \mathbb{A}^1,$$
$$f : U \to X,$$
$$f(\alpha_1, \alpha_2) := \alpha_2/\alpha_1.$$

This map assigns to a point P not contained in the vertical axis $V(x_1)$ the slope of the line $\overline{OP}$ through P and the origin $O := (0, 0)$. We claim that there does *not* exist a larger open subset

$$U \subsetneq V \subset Y$$

and a morphism

$$F : V \to X$$

that extends f, i.e., for which $F|_U = f$. Suppose otherwise that such an F exists. Define a new map

$$g : V \to \mathbb{P}^1$$

by composing F with the inclusion $X \hookrightarrow \mathbb{P}^1$, and denote by ∞ the unique element of $\mathbb{P}^1 - X$. Set $O := (0, 0)$, and let

$$\pi : \mathbb{A}^2 - \{O\} \to \mathbb{P}^1$$

be as in the previous section. Then π agrees with g on the nonempty open $V \cap (\mathbb{A}^2 - \{O\})$, which shows that $g(0, \beta) = \infty$ for all nonzero β for which $(0, \beta)$ belongs to V. But $F(0, \beta) \neq \infty$ for all such β because F takes values in $X = \mathbb{A}^1$, giving the required contradiction.

13.4 How we will abbreviate the arguments in the above examples in what follows

The arguments of the previous two sections will be abbreviated thusly:

1. The morphism

$$\mathbb{A}^2 - \{(0,0)\} \ni (\alpha_1, \alpha_2) \mapsto [\alpha_1, \alpha_2] \in \mathbb{P}^1$$

   does not extend to a morphism $f : \mathbb{A}^2 \to \mathbb{P}^1$, because if it did, then for each $\beta \in k$ the quantity $f(0,0)$ would coincide with the limiting value of as $\gamma \to 0$ with $\gamma \neq 0$ of $f([\gamma, \beta\gamma]) = [1, \beta]$, whence $f(0,0) = [1, \beta]$; since the LHS is independent of β but the RHS is not, we derive a contradiction.

2. The morphism

$$\mathbb{A}^2 - V(x_1) \ni (\alpha_1, \alpha_2) \mapsto \alpha_1/\alpha_2 \in \mathbb{A}^1$$

   does not extend to a morphism $f : U \to \mathbb{A}^1$ for any open set U properly containing $\mathbb{A}^2 - V(x_1)$. Indeed, suppose otherwise that it did. Choose a point $(0, \beta)$ with $\beta \neq 0$ belonging to the nonempty open subset $U \cap V(x_1)$ of the vertical axis $V(x_1)$. Then the limit as $\gamma \to 0$ with $\gamma \neq 0$ of

$$f(\gamma, \beta) = \beta/\gamma$$

   is obviously ∞, which belongs to $\mathbb{P}^1$ but not to $\mathbb{A}^1$. Therefore no such extension f having codomain $\mathbb{A}^1$ exists.

13.5 Graphs are closed

Let X, Y be varieties and $f : Y \to X$ a morphism between them. The *graph* of f is the subset

$$\Gamma_f := \{(\alpha, f(\alpha)) : \alpha \in X\} \subset Y \times X$$

of the product variety. Since X is separated, the graph Γ_f is always a closed subvariety of $Y \times X$, being the equalizer $\Gamma_f = \operatorname{eq}(f_1, f_2)$ of the morphisms $f_1, f_2 : Y \times X \to X$ given by

- $f_1(\alpha, \beta) := f(\alpha)$, and
- $f_2(\alpha, \beta) := \beta$.

14 Basic properties of dimension

14.1 Todo stuff

TODO: defn, etc. If $X = \cup U_\alpha$ then $\dim_X = \sup \dim U_\alpha$.

If X is an irreducible affine variety and $U \subset X$ is a nonempty open, then $\dim X = \dim U$.

14.2 Krull's Hauptidealsatz

15 Products of varieties

15.1 Overview

The word "products" in the title refers to finite products in one of the categories we have considered thus far. We briefly recall what this means. Let $\mathcal{C}$ be a category, such as

- the category of affine varieties with polynomial maps,
- the category of k-spaces,
- the category of prevarieties,
- the category of varieties, or
- the category of quasi-projective varieties,

and let, X_1, X_2 be two objects in $\mathcal{C}$. Recall that a *categorical product* of X_1 and X_2 is an object in $\mathcal{C}$, denoted $X_1 \times X_2$, which comes equipped with morphisms $p_i : X_1 \times X_2 \to X_i$ $(i = 1, 2)$ (called *projection maps*) which are universal in the sense that any object Z and pair of morphisms $f_i : Z \to X_i$ $(i = 1, 2)$ factors through them in the sense that there exists a unique morphism denoted $f_1 \times f_2 : Z \to X_1 \times X_2$ which forms a commutative diagram with everything in sight. Any two categorical products $X_1 \times X_2$ are isomorphic, and there is a unique isomorphism connecting them which commutes with the projection maps. (Consult google/wikipedia/???.)

I will be very brief in this section; please consult the course references, which discuss this stuff very well.

15.2 Products of affine varieties exist and are given by taking the tensor product of affine coordinate rings

We saw on the homework that products exist in the category of affine varieties: if

$$X \subset \mathbb{A}^m = \operatorname{Specm} k[x_1, \ldots, x_m]$$

and

$$Y \subset \mathbb{A}^n = \operatorname{Specm} k[y_1, \ldots, y_n]$$

are given by $X = V(\mathfrak{a})$ and $Y = V(\mathfrak{b})$, then the affine variety $X \times Y \subset \mathbb{A}^{m+n}$ cut out by the ideal

$$\mathfrak{a} \otimes \mathfrak{b} \subset k[x_1, \ldots, x_m] \otimes_k k[y_1, \ldots, y_n] \cong k[x_1, \ldots, x_m, y_1, \ldots, y_n]$$

is a categorical product of X and Y. Moreover, the last isomorphism induces a natural identification

$$A(X \times Y) \cong A(X) \otimes_k A(Y).$$

Note (cf. homework) that the set underlying $X \times Y$ is the product of the sets underlying X and Y, but that the topologies are distinct.

15.3 Products of prevarieties exist and are given by glueing the products of affine varieties

The construction of products of affine varieties given above is "sufficiently functorial" that it can be glued together to show that products of arbitrary prevarieties exist; see the relevant section of Hartshorne II.3 (replace "scheme" with "prevariety" and take $S := k$) or somewhere in the notes of Milne/Gathmann.

15.4 Products of varieties are varieties

Given a pair of varieties X, Y, their product exists as a prevariety $X \times Y$ by the result of the previous section. It is not hard to verify (using the definition of separatedness and the functorial properties of the product) that $X \times Y$ is itself separated, hence defines a variety.

15.5 Products of quasi-projective varieties are quasi-projective and described by the Segre embedding

Given a pair of quasi-projective variety $X \subset \mathbb{P}^m, Y \subset \mathbb{P}^n$ in coordinates $x_0, \ldots, x_m, y_0, \ldots, y_n$, the product $X \times Y$ exists as an abstract variety, but it is not immediately clear that it may be realized as a quasi-projective variety. The construction furnishing this realization is the *Segre embedding*: with the integer $N \in \mathbf{Z}_{\geq 0}$ determined by the relation

$$N + 1 = (m + 1)(n + 1)$$

and with $\mathbb{P}^N$ equipped with variables z_{ij} taken over the indices $i \in \{0..m\}$ and $j \in \{0..n\}$ (and perhaps thought of as indexing the rows and columns of an $(m + 1) \times (n + 1)$ matrix), the Segre embedding is defined initially to be the *set-theoretic* map

$$s : \mathbb{P}^m \times \mathbb{P}^n \to \mathbb{P}^N$$

$$[\alpha_0, \ldots, \alpha_m] \times [\beta_0, \ldots, \beta_n] \mapsto [\ldots, \alpha_i \beta_j, \ldots]$$

from the product *set* $\mathbb{P}^m \times \mathbb{P}^n$ to the projective space $\mathbb{P}^N$. This map is verified to be injective with closed image cut out by the equations $z_{ij} z_{kl} = z_{il} z_{kj}$ taken over all relevant indices. It thereby allows one to identify the product set $\mathbb{P}^m \times \mathbb{P}^n$ with a closed subvariety of $\mathbb{P}^N$. This identification is in turn used to *define* the varietal structure on $\mathbb{P}^m \times \mathbb{P}^n$. Then $\mathbb{P}^m \times \mathbb{P}^n$ may be verified to be a categorical product. One likewise identifies $X \times Y$ with its image under s and verifies that as a quasi-projective variety in $\mathbb{P}^N$, it defines a categorical product of X and Y.

It's worth remarking that closed subsets of $\mathbb{P}^m \times \mathbb{P}^n$ are those defined by equations $f = 0$ where f is a bihomogeneous polynomial in the coordinate variables, i.e., $f \in k[x_0, \ldots, x_m, y_0, \ldots, y_n]$ is homogeneous of one degree in the variables x_i and of homogeneous some possibly different degree in the variables y_j. (Note that any closed subset of $\mathbb{P}^m \times \mathbb{P}^n$ arises from homogeneous polynomial equations on the variables $z_{ij} = x_i y_j$ under the Segre embedding into $\mathbb{P}^N$.)

16 Rational maps

There is a fair bit of vocabulary introduced in this section, but we should perhaps warn the reader that none of the results are difficult or deep in any way; they essentially amount to syntactic sugar for systematically throwing away "small" subsets of the spaces under consideration.

16.1 Definition

Let X, Y be varieties. A *rational map*

$$f : Y - - > X$$

is an equivalence class of pairs (U, f_U) where U is a nonempty open subset of Y and $f_U : U \to Y$ is a morphism, with two such pairs identified if they agree on a common dense open subset of their overlap. Thus

$$(U_1, f_{U_1}) \sim (U_2, f_{U_2})$$

if and only if there exists an open subset $W \subset U_1 \cap U_2$ that is dense[17] and for which

$$f_{U_1}|_W = f_{U_2}|_W.$$

This notion defines an equivalence relation, with the only nontrivial verification being that of transitivity: if

$$(U_1, f_{U_1}) \sim (U_2, f_{U_2}) \sim (U_3, f_{U_3})$$

and

$$W \subset U_1 \cap U_2,$$
$$W' \subset U_2 \cap U_3$$

are dense open subsets as above, then $W \cap W'$ is a dense open subset of $U_1 \cap U_3$ (see Section 12.1) on which f_{U_1} and f_{U_3} coincide, as required.

We shall abuse notation by writing f both for the rational map $f : Y - - > X$ and for a morphism $f : U \to X$ representing f.

[17] either in $U_1 \cap U_2$ or in X; the two notions are the same, see Section 12.1.2

16.2 Strengthening of definition to requiring agreement on all of overlap

The definition of equivalence may be strengthened somewhat as follows: two pairs (U, f) and (V, g), with U, V dense open subsets of Y and $f : U \to X$ and $g : V \to X$ morphisms, represent the same rational map $Y - - > X$ if and only if

$$f|_{U \cap V} = g|_{U \cap V}.$$

The latter condition clearly implies $(U, f) \sim (V, g)$. Conversely, if W is a dense open subset of $U \cap V$ on which f and g coincide, then because X is separated (see Section 13), we deduce that $f = g$ holds on $U \cap V$, as required.

16.3 The domain of definition

16.3.1 Definition

Let $f : Y - - > X$ be a rational map between varieties. Say that f is *defined* (or perhaps *regular*) at a point $\beta \in Y$ if there exists a representative (U, f_U) for f with the property that U contains β. The set of points in Y at which f is defined is called the *domain of definition* of f. It is a dense open subset of f, given explicitly as the union $U := \cup U_\alpha$ taken over all representatives (U_α, f_{U_α}) for f. Since "being a morphism" is local, the rational map f is actually represented by a morphism $f : U \to X$ on its domain of definition U, that is to say, U arises as one of the U_α.

16.3.2 Example: parametrization of the circle

Set

$$X := V(x_1^2 + x_2^2 - 1) \subset \mathbb{A}^2$$

and

$$Q := (1, 0) \in X.$$

Consider the rational map

$$f : X - - > \mathbb{P}^1$$

$$(\alpha_1, \alpha_2) \mapsto [\alpha_1 - 1, \alpha_2]$$

sending a point $P \in X$ to the slope of the line $\overline{PQ}$. It is clear that f is defined at a point $P \in X$ whenever $P \neq Q$. Thus, we should interpret the above formula as defining f to be the rational map represented on the complement

$$U := X - \{Q\}$$

of Q in X by the morphism

$$U \to \mathbb{P}^1$$

$$(\alpha_1, \alpha_2) \mapsto [\alpha_1 - 1, \alpha_2].$$

Note that if one formally takes $P = Q$, then the point "$[1 - 1, 0] = [0, 0]$" of projective space does not make sense. Let us compute the domain of definition of f; call it D, note that $D \supset X - \{Q\}$, and write

$$f : D \to \mathbb{P}^1.$$

It seems reasonable to suspect that f may be extended to a morphism $X \to \mathbb{P}^1$, i.e., that $D = X$, since as P tends toward Q along the variety X, the line $\overline{PQ}$ becomes progressively vertical. To see this, note that for α in X, one has

$$(\alpha_1 - 1)(\alpha_1 + 1) + \alpha_2^2 = 0,$$

whence

$$[\alpha_1 - 1, \alpha_2] = [-\alpha_2, \alpha_1 + 1]$$

whenever both sides make sense, i.e., whenever $P \neq Q$ and $P \neq R := (-1, 0)$. Therefore the identity

$$f(\alpha) = [-\alpha_2, \alpha_1 + 1]$$

gives a representative morphism for f on $X - \{R\}$, while the original definition gives a representative morphism for f on $X - \{Q\}$; together, they glue to define f as a morphism on all of X, whence $D = X$, as claimed.

16.3.3 Example: the quotient map defining the projective line

Set $O := (0, 0)$ and consider the morphism

$$\pi : \mathbb{A}^2 - \{O\} \to \mathbb{P}^1$$

$$(\alpha_1, \alpha_2) \mapsto [\alpha_1, \alpha_2].$$

We may think of it as a rational map

$$\pi : \mathbb{A}^2 - - > \mathbb{P}^1$$

defined on $\mathbb{A}^2 - \{O\}$. The domain of definition of π is $\mathbb{A}^2 - \{O\}$, i.e., π cannot be extended to a morphism $\mathbb{A}^2 \to \mathbb{P}^1$. Indeed, this assertion was verified already in Section 13.2.

16.3.4 Example: something else

Similarly, the morphism $D_{\mathbb{A}^2}(x_1) \ni (\alpha_1, \alpha_2) \mapsto \alpha_2/\alpha_1 \in \mathbb{A}^1$ can be thought of as the rational map

$$f : \mathbb{A}^2 - - > \mathbb{A}^1$$

$$(\alpha_1, \alpha_2) \mapsto \alpha_2/\alpha_1.$$

The result of Section 13.3 translates to the assertion that the domain of definition of f is the complement $D_{\mathbb{A}^2}(x_1)$ of the vertical axis.

16.3.5 Example: cuspidal cubic

Set

$$X := V(x_1^3 - x_2^2) \subset \mathbb{A}^2.$$

The rational map

$$f : X - - > \mathbb{A}^1$$

$$(\alpha_1, \alpha_2) \mapsto \alpha_2/\alpha_1$$

is defined initially on $X - \{O\}$ with $O := (0,0) = X \cap V(x_2)$.

Now, consider the morphism

$$p : \mathbb{A}^1 := \operatorname{Specm} k[t] \to X$$

$$\gamma \mapsto (\gamma^2, \gamma^3) \in X.$$

We saw on some homework set that this morphism is a bijective homeomorphism that is not an isomorphism of varieties. Nevertheless, we may use it to set-theoretically parametrize X. The subset $\mathbb{A}^1 - \{0\}$ corresponds under p with $X - \{O\}$.

If we pull f back under p, we get the rational map

$$\mathbb{A}^1 - \{0\} \to \mathbb{A}^1$$

$$\gamma \mapsto f(p(\gamma)) = \gamma^3/\gamma^2 = \gamma.$$

This morphism extends to $\gamma = 0$ where it is given by $0 \mapsto 0$.

Thus, it might seem intuitively plausible for the rational map f to have domain of definition all of X, with its extension to O given by $f(O) := 0$. (Draw a picture of what's happening here at the level of real points.) *But that's not true.* The issue is that there are "not enough" regular functions on X. The domain of definition of f is in fact $X - \{O\}$, that is to say, it is not defined at O. Indeed, suppose otherwise that there exists a morphism $f : X \to \mathbb{A}^1$ (denoted also by f by abuse of notation) that extends the given morphism $X - \{O\} \to \mathbb{A}^1$. Then, by what we saw ages ago, f is represented as the restriction of some polynomial F in the coordinate functions x_1, x_2. Recall that

$$p^\sharp(F) \in k[t]$$

denotes the polynomial representing the pullback $p^\sharp(F) := F \circ p$ of F under the parametrization p from above. Note that

$$p^\sharp(x_1) = t^2$$

and

$$p^\sharp(x_2) = t^3$$

whence that

$$p^\sharp(F) \in \operatorname{image}(p^\sharp) = k[t^2, t^3] \subset k[t].$$

For any $\gamma \in \mathbb{A}^1 - \{0\}$ we have

$$p^\sharp(F)(\gamma) = f(p(\gamma)) = \gamma.$$

Since $p^\sharp(F)$ is a polynomial, it follows that

$$p^\sharp(F) = t.$$

We have thus deduced that

$$t \text{ belongs to } k[t^2, t^3],$$

which gives the required contradiction.

16.4 Dominance

A rational map $f : Y - - > X$ is called *dominant* if it satisfies either of the following equivalent conditions:

- For some representative morphism $f : U \to X$, the image $f(U)$ is dense in X.
- For *every* representative morphism $f : U \to X$, the image $f(U)$ is dense in X.

The equivalence of these conditions was verified in class and is left here as an exercise. An example of a rational map that is *not* dominant is the morphism $f : \mathbb{A}^1 \to \mathbb{A}^1$ given by $f(\alpha) := \gamma$ for some fixed $\gamma \in k$.

Note that any rational map $f : Y - - > X$ induces a dominant rational map

$$f : Y - - > Z$$

where

$$Z := \overline{f(Y)}$$

is the closure of the image. Therefore there is not much loss of generality in considering dominant rational maps.

Exercise 32. Suppose X, Y are irreducible varieties and

$$f : Y - - > X$$

is a rational map. Then f is dominant if and only if for any nonempty affine open subsets $U \subset X$ and $V \subset Y$ with V contained in $f^{-1}(U)$ and also contained in the domain of definition of f, the induced morphism of affine varieties

$$f|_V : V \to U$$

induces a morphism of affine coordinate rings

$$f|_V^\sharp : A(U) \to A(V)$$

that is injective, i.e., with the property that

$$f|_V^{\sharp\,-1}(0) = \{0\}.$$

(See Section 6.9.)

16.5 Dominant rational maps of irreducible varieties can be composed

Given rational maps $f : Y - - > X$ and $g : Z - - > Y$, when can they be composed to form something like a rational map

$$f \circ g : Z - - > X?$$

Well, let $W \subset Z$ and $V \subset Y$ be domains of definition for g and f, respectively. It is then natural to attempt to represent $f \circ g$ by the pair

$$(U, f \circ g|_U)$$

where

$$U := W \cap g^{-1}(V)$$

is the intersection of the domain of g with the preimage thereunder of the domain of f. In order for this pair to define a rational map, it is necessary that

$$\text{the set } U \text{ is dense and open in } Z.$$

If that is the case, we say that the composition $f \circ g$ is defined. It then gives a rational map which is independent of the choice of representatives for the rational maps f, g.

However, the condition (16.5) need not hold in general. For instance, if one takes g to be a constant function taking its value outside of V, then $g^{-1}(V) = \emptyset$.

It is convenient to impose "reasonable" conditions on the spaces X, Y, Z and the rational maps f, g that imply (without much fuss) that the composition $f \circ g$ is defined. If we suppose that g is dominant, then at least $g^{-1}(V)$ is a nonempty open subset of Z, but it need not be dense in general; to ensure the latter, we might as well take Z to be irreducible, so that any nonempty open is dense. Thus $f \circ g$ is defined whenever Z is irreducible and g is dominant.

In particular, we can always compose dominant rational maps between irreducible varieties, and their composition is associative, hence defines the structure of a category.

16.6 Function fields and stalks

16.6.1 Definition

Let X be an irreducible variety. The set of rational maps $X - - > \mathbb{A}^1$ is denoted $k(X)$ and called the *function field* of X. It is, in fact, a field that contains k:

- Any constant function $X \ni \alpha \mapsto \gamma \in k$ defines a rational function.
- Given $f, g \in k(X)$, we may define the sum $f + g \in k(X)$ by adding the functions elementwise on the (nonempty, hence open and dense) intersection of their domains of definition. We may similarly define the product fg.

- Let $f \in k(X)$ be nonzero with domain of definition U. Then $D_X(f)$ is nonempty, hence open and dense; defining $1/f$ to be the rational function represented by the pair $(U \cap D_X(f), 1/f)$, we obtain a multiplicative inverse for f inside $k(X)$.

We carry over to rational functions much of the same terminology that was used to discuss rational maps. In particular, we may speak of the domain of definition U of a rational function $f \in k(X)$, which is the largest open subset of X with the property that f is represented on U by a morphism $f : U \to \mathbb{A}^1$, or equivalently, by a regular function $f \in \mathcal{O}_X(U)$.

16.6.2 Preservation under passing to nonempty open subsets

It is important to note that if X is an irreducible variety and U is a nonempty (hence dense) open subset of X, regarded as a (necessarily irreducible) variety with the induced structure, then there is a natural identification $k(X) = k(U)$: given a rational function $f \in k(X)$ represented by a morphism $f : V \to \mathbb{A}^1$ for some nonempty open $V \subset X$, its restriction $f|_{U \cap V}$ defines a rational function on U. Similarly, given a rational function $f \in k(U)$ represented by a morphism $f : V \to \mathbb{A}^1$ for some nonempty open $V \subset U$, the same morphism defines a rational function on X.

16.6.3 Regular functions on open subsets as rational functions

For any open subset $U \subset X$, there is a natural identification of the ring $\mathcal{O}(U)$ of regular functions $f : U \to k$ (or equivalently, morphisms $f : U \to \mathbb{A}^1$) as a subring of the function field $k(X)$.

We recover $\mathcal{O}(U)$ as the set of all rational functions $f \in k(X)$ which are defined at every ponit of U. This definition is computationally useful.

16.6.4 Computation in the affine case

If X is an irreducible affine variety with affine coordinate ring $A := A(X)$, then A is an integral domain, and so the natural inclusion $A = \mathcal{O}_X(X) \hookrightarrow k(X)$ extends to a k-equivariant embedding of fraction fields $\mathrm{Frac}(A) \hookrightarrow k(X)$. This map is actually an isomorphism, that is to say, there is a natural identification (of field extensions of k)

$$\mathrm{Frac}(A) = k(X).$$

To see this, it remains only to check that each $f \in k(X)$ can be represented as a ratio of elements of A. Let $D_X(a)$, $a \in A$ be a nonempty basic open subset of the domain of definition of f. Then $f|_{D_X(a)} = g/a^N$ for some $g \in A$ and some $N \geq 0$. Therefore f agrees on the nonempty open set $D_X(a)$ with the ratio g/a^N of elements of A, as required.

By combining this result with that of Section 16.6.2, we see that for any irreducible variety X, the function field $k(X)$ may be computed as $\mathrm{Frac}(A(U))$ for any (nonempty) affine open subset U of X. Since such an affine open subset always exists, we deduce that the function field $k(X)$ of any irreducible variety is a finitely-generated k-algebra.

Example 81. $k(\mathbb{A}^n) = k(x_1, \dots, x_n)$. $k(\mathbb{P}^n) \cong k(x_1/x_0, \dots, x_n/x_0)$.

16.6.5 Stalks

More generally, let X be any variety (not necessarily irreducible). For each irreducible subvariety Z of X, i.e., irreducible closed subset $Z \subset X$, denote by $\mathcal{O}_{X,Z}$ the set of equivalence classes of pairs (U, f), where $U \subset X$ is an open set for which $U \cap Z \neq \emptyset$ and $f \in \mathcal{O}_X(U)$ is a regular function on U, with two such pairs (U, f) and (V, g) deemed equivalent if either of the following equivalent conditions hold:

- There exists an open set $W \subset U \cap V$ for which $W \cap Z \neq \emptyset$ and $f = g$ holds on W.
- $f = g$ holds on $U \cap V$.

(The proof of the equivalence of these conditions is, as before, as consequence of the separatedness of $\mathbb{A}^1$.) The set $\mathcal{O}_{X,Z}$ has the natural structure of a k-algebra. It is called the *local ring* of X at the irreducible subvariety Z.

We have considered some special cases before:

- When $Z = \{\alpha\} \subset X$ is a point, $\mathcal{O}_{X,\{\alpha\}}$ is the stalk $\mathcal{O}_{X,\alpha}$ at the point α as considered in Exercise 27.
- When X is irreducible and $Z = X$ is the entire space, then $\mathcal{O}_{X,X}$ is the function field $k(X)$.

In general, the computation of local rings reduces to the affine case: if X, Z are as above and U is a (nonempty) affine open subset of X for which $U \cap Z \neq \emptyset$, then one has a natural identification

$$A(U)_{\mathfrak{p}_Z} = \mathcal{O}_{X,Z}$$

where $\mathfrak{p}_Z$ is the prime ideal of $A(U)$ corresponding to the irreducible subset $U \cap Z$ of U. The verification is as in the previous section and left to the reader.

For irreducible subvarieties Z_1, Z_2 of X with $Z_1 \subset Z_2$, there is a natural inclusion map

$$\mathcal{O}_{X,Z_1} \hookrightarrow \mathcal{O}_{X,Z_2}$$

given on representatives by the identity.

When X is itself irreducible, we can think of every local ring $\mathcal{O}_{X,Z}$ for Z an irreducible subvariety of X, and in particular, every local ring $\mathcal{O}_{X,\alpha}$ for $\alpha \in X$ a point, as being contained inside the function field $k(X)$. In the affine case, this corresponds to thinking of the local rings of an integral domain as all being contained inside the fraction field.

16.6.6 Pullback of rational functions under dominant rational maps

Given irreducible varieties X, Y and a dominant rational map $f : Y - - > X$ and a rational function $g \in k(X)$, the composition $g \circ f$ defines a rational map $Y - - > \mathbb{A}^1$, hence a rational function

$$f^\sharp(g) := g \circ f \in k(Y).$$

Thus to each dominant rational map $f : Y - - > X$ one obtains a k-algebra morphism

$$f^\sharp : k(X) \to k(Y)$$

of function fields via pullback.

Conversely, given a k-algebra morphism

$$\phi : k(X) \to k(Y)$$

of function fields, one obtains a dominant rational map

$$\phi^\flat : Y - - > X$$

given by:

- taking some (nonempty) affine open subset $U \subset X$;
- identifying U with an affine variety $U \subset \mathbb{A}^n$, with affine coordinate ring $A(U)$ generated by the images $\overline{x}_1, \ldots, \overline{x}_n$ of the coordinate functions $x_1, \ldots, x_n$ on $\mathbb{A}^n$, say;
- taking the images

  $$\psi_1 := \phi(\overline{x}_1) \in k(Y), \quad \ldots, \quad \psi_n := \phi(\overline{x}_n) \in k(Y)$$

  of the generators, thinking of them as rational functions on Y represented by morphisms

  $$\psi_1 : V_1 \to k,$$
  $$\cdots,$$
  $$\psi_n : V_n \to k$$

  for some nonempty open subsets $V_1, \ldots, V_n$ of Y; and, finally,
- taking the intersection $V := V_1 \cap \cdots \cap V_n$ (a nonempty open subset of Y) and defining $\phi^\flat$ to be the rational function represented by the morphism

  $$\phi^\flat : V \to U \subset \mathbb{A}^n$$

  $$\beta \mapsto \phi^\flat(\beta) := (\psi_1(\beta), \ldots, \psi_n(\beta)).$$

The verification that $\phi^\flat$ takes values where we say it does is as in Section 6.3; moreover, $\phi^\flat$ is independent of the choice of affine open subset U. Similarly, one finds that
$$(f^\sharp)^\flat = f$$
and that
$$(\phi^\flat)^\sharp = \phi$$
for all such f, ϕ. We obtain in this way mutually inverse bijections
$$\{ \text{ dominant rational maps } Y - - > X\} \cong \mathrm{Hom}_k(k(X), k(Y)).$$
This identification is very handy. It lets us say things like "consider the rational map
$$\mathbb{A}^1 = \operatorname{Specm} k[y_1] \to \mathbb{A}^1 = \operatorname{Specm} k[x_1]$$
corresponding to the morphism of function fields
$$x_1 \mapsto \frac{y_1 + 1}{y_1 - 1}."$$

One can show (compare with the discussion of Section 5.5) that every field of finite transcendence degree over k arises as the function field of some irreducible (affine) variety X. A consequence of the bijection just established is that the functor $X \mapsto k(X)$, $f \mapsto f^\sharp$ defines an equivalence of categories
$$\{ \text{ irreducible varieties with dominant rational maps } \} \to \{ \text{ fields of finite transcendence degree over } k \ \}.$$

16.7 How to write one down in practice

Let Y be an irreducible variety and $X \subset \mathbb{P}^n$ a quasi-projective variety. Then every rational map
$$f : Y - - > X$$
is of the form
$$f = [f_0, \dots, f_n]$$
$$\beta \mapsto [f_0(\beta), \dots, f_n(\beta)]$$
for some rational functions $f_0, \dots, f_n \in k(Y)$ with the properties:

- some f_i is not identically 0, and
- for all β for which some $f_i(\beta) \neq 0$, the point $[f_0(\beta), \dots, f_n(\beta)]$ belongs to X.

Conversely, each such datum defines a rational map. One can check that f is regular at a point $\beta \in Y$ if there exists $g \in k(Y)^\times$ for which each gf_i is regular at β and some $gf_i(\beta) \neq 0$; the choice of g may depend upon β. (See Section 16.3.2 for an example along these lines.)

If moreover $Y \subset \mathbb{P}^m$ is quasi-projective, then each rational map $f : Y - - > X$ may be written $f = [f_0, \dots, f_n]$ where the f_i are homogeneous polynomials of the same degree with the property that some $f_i \notin I_{\mathbb{P}^m}(Y)$ and with the property that each homogeneous polynomial h on $\mathbb{P}^n$ that vanishes on X pulls back under f to a homogeneous polynomial $h \circ f$ on $\mathbb{P}^m$ that vanishes on Y, i.e.,

$$h(f_0(y_0, \ldots, y_m), \ldots, f_n(y_0, \ldots, y_m)) \in I_{\mathbb{P}^m}(Y) \text{ for all homogeneous } h \in I_{\mathbb{P}^n}(X).$$

The rational map f is defined at a point $\beta \in Y$ iff there exist homogeneous polynomials $g_0, \ldots, g_n$ of the same degree so that each $f_i g_j \equiv f_j g_i \pmod{I(Y)}$ and some $g_i(\beta) \neq 0$; in that case, $f(\beta) = [g_0(\beta), \ldots, g_n(\beta)]$.

If Y is a variety with irreducible decomposition $Y = \cup_i Y_i$ and X is any variety, then a rational map $Y - - > X$ is the same as a tuple of rational maps $Y_i - - > X$. Thus we may often reduce to the irreducible case and work in the function field.

A *morphism* $Y \to X$ may now be re-defined as a rational map which is regular at each point. This definition is computationally useful.

16.8 Birational equivalence and isomorphism of open subsets

Two varieties X, Y will be called *birationally equivalent* (or one will say that X is *birational to* Y, etc.) if there exist dominant rational maps $f : X \to Y$ and $g : Y \to X$ whose compositions $g \circ f, f \circ g$ are defined and coincide with the identity maps $1_X, 1_Y$. A rational map $f : Y - - > X$ will be called *birational* if it is dominant and admits a dominant rational inverse.

For example, one can verify (as was done in lecture) that the rational maps of the examples of Sections 16.3.2 and 16.3.5 are birational; the second of these examples, however, is not an isomorphism.

Let X, Y be irreducible varieties. Then the following are equivalent:

- X is birational to Y.
- $k(X)$ is isomorphic as a k-algebra to $k(Y)$.
- There exist nonempty open subsets $X_1 \subset X, Y_1 \subset Y$ which are isomorphic: $X_1 \cong Y_1$.

The results of Section 16.6.6 imply the equivalence of the first two conditions, and the third condition clearly implies the first, so it remains only to verify that if there exist nonempty open subsets $X_0 \subset X, Y_0 \subset Y$ and morphisms

$$f : X_0 \to Y$$

and

$$g : Y_0 \to X$$

with the property that the identities $g \circ f = 1_X$ and $f \circ g = 1_Y$ hold where defined, then there exist nonempty open subsets $X_1 \subset X, Y_1 \subset Y$ which are isomorphic. Our hypotheses imply that $g \circ f$ is defined on $f^{-1}(Y_0)$ and hence coincides with the identity 1_X there; similarly, $f \circ g$ is defined on $g^{-1}(X_0)$ and coincides with the identity 1_Y there. Set

$$X_1 := f^{-1}(g^{-1}(X_0))$$

and

$$Y_1 := g^{-1}(f^{-1}(Y_0)).$$

If α belongs to X_1, then $f(\alpha)$ belongs to $g^{-1}(X_0)$, whence $f(g(f(\alpha))) = f(\alpha) \in Y_0$. Thus $g(f(\alpha)) \in f^{-1}(Y_0)$ and hence $f(\alpha) \in g^{-1}(f^{-1}(Y_0)) = Y_1$, so that f induces a morphism $f : X_1 \to Y_1$. Similarly, g induces a morphism $g : Y_1 \to X_1$. These morphisms are mutually inverse since $X_1 \subset f^{-1}(Y_0)$ and $Y_1 \subset g^{-1}(X_0)$.

17 Blowups

17.1 At an r-tuple of regular functions

17.1.1 Definition

Take an affine variety $X \subset \mathbb{A}^n$, and let $f_1, \dots, f_r \in A(X)$ be some r-tuple of regular functions for a natural number r. Denote by

$$U := X - V_X(f_1, \dots, f_r)$$

the open subset of X consisting of those points α for which there exists an i so that $f_i(\alpha) \neq 0$. We may then define a morphism

$$f := [f_1, \dots, f_r] : U \to \mathbb{P}^{r-1}$$

by the formula

$$f(\alpha) := [f_1(\alpha), \dots, f_r(\alpha)].$$

The most interesting case for us is when U is dense in X, as we shall henceforth assume; in other words, we assume that $V(f_1, \dots, f_r)$ contains no irreducible component of X. We may then think of f as a rational map $f : X - - > \mathbb{P}^{r-1}$. It can happen that f does *not* extend to a morphism $X \to \mathbb{P}^{r-1}$. A simple example in which this happens is given (as verified earlier in Section 13.2) by the natural projection map

$$\mathbb{A}^2 - \{0\} \to \mathbb{P}^1$$

$$(\alpha_1, \alpha_2) \mapsto [\alpha_1, \alpha_2].$$

One *can* always extend f to a certain *cover* $\widetilde{X}$ of X, as follows: Denote by

$$\Gamma_f := \{(\alpha, f(\alpha)) : \alpha \in U\} \subset U \times \mathbb{P}^{r-1} \subset \mathbb{A}^n \times \mathbb{P}^{r-1}$$

the graph of the morphism f. Recall from Section 13.5 that Γ_f is closed in the product space $U \times \mathbb{P}^{r-1}$. Note however, as in the above example, that Γ_f need not be closed in the larger product space $X \times \mathbb{P}^{r-1}$.

Definition 82. The *blow-up* $\widetilde{X}$ of X at $(f_1, \ldots, f_r)$ is the closure of Γ_f inside $X \times \mathbb{P}^{r-1}$. It is a variety, and comes equipped with a natural morphism

$$\pi : \widetilde{X} \to X$$
$$(\alpha, \beta) \mapsto \alpha$$

given by projection onto the first factor.

17.1.2 Birationality

Since Γ_f is closed in $U \times \mathbb{P}^{r-1}$, we have $\widetilde{X} \cap (U \times \mathbb{P}^{r-1}) = \Gamma_f$, whence that the map

$$\pi|_{U \times \mathbb{P}^{r-1}} : \Gamma_f \to U$$
$$(\alpha, \beta) = (\alpha, f(\alpha)) \mapsto \alpha$$

is an isomorphism with inverse

$$U \to \Gamma_f$$
$$\alpha \mapsto (\alpha, f(\alpha)).$$

In this way we identify U with the open subset Γ_f of $\widetilde{X}$. In particular, $\widetilde{X}$ and X contain isomorphic dense open subsets identified under π, and hence are birational to one another under the birational equivalence π.

If we use coordinates $x_1, \ldots, x_n, y_1, \ldots, y_r$ on $\mathbb{A}^n \times \mathbb{P}^{r-1}$, then $\pi^{-1}(U) = \Gamma_f \cong U$ is the subset of $U \times \mathbb{P}^{r-1}$ on which the polynomial equations

$$y_i f_j = y_j f_i \tag{16}$$

hold. By the definition of Zariski closure, the same equations hold on $\widetilde{X}$. The complement of $U \cong \Gamma_f$ inside $\widetilde{X}$ is given by $\pi^{-1}(X - U) = \widetilde{X} - U$, called the *exceptional set*, and denoted

$$E := \pi^{-1}(X - U).$$

The equations (16) hold on the exceptional set, but do not in general define it.

17.1.3 Compatibility with passing to closed subvarieties

Suppose Y is a closed subvariety of X for which $Y \cap U$ is also dense in Y.[18] then the natural inclusion

$$Y \cap U \hookrightarrow U$$

induces an inclusion of graphs

$$\Gamma_{f|_{Y \cap U}} = \Gamma_f \cap (Y \cap U) \times \mathbb{P}^{r-1} \hookrightarrow \Gamma_f$$

which upon taking closures inside $X \times \mathbb{P}^{r-1}$ gives a natural identification

$$\widetilde{Y} = \text{ closure of } \Gamma_f \cap (Y \cap U) \times \mathbb{P}^{r-1} \text{ inside } X \times \mathbb{P}^{r-1}.$$

[18] One can dispense with this assumption; we impose it because it holds in the examples of immediate interest.

17.1.4 Canonicity

The cover $\pi : \widetilde{X} \to X$ attached as above to the r-tuple $f_1, \dots, f_r$ actually depends only (up to isomorphism) upon the ideal $\mathfrak{a} := (f_1, \dots, f_r)$. To see this, suppose given some other set of generators $\mathfrak{a} = (g_1, \dots, g_s)$. Denote by $\pi' : \widetilde{X}' \to X$ the cover they give rise to. Thus if we write $f := [f_1, \dots, f_r] : U \to \mathbb{P}^r$ and $g := [g_1, \dots, g_s] : U \to \mathbb{P}^s$, then $\widetilde{X}$ is the closure in $X \times \mathbb{P}^r$ of $\pi^{-1}(U) = \{(\alpha, f(\alpha)) : \alpha \in U\}$ and $\widetilde{X}'$ is likewise the closure in $X \times \mathbb{P}^s$ of $\{(\alpha, g(\alpha)) : \alpha \in U\}$; note that $X - U = V(\mathfrak{a}) = V(f_1, \dots, f_r) = V(g_1, \dots, g_s)$. Our aim is to define a map $\kappa : \widetilde{X} \to \widetilde{X}'$ which is compatible with the projections in the sense that $\pi' \circ \kappa = \pi$. Any such map κ must send $(\alpha, f(\alpha))$ to $(\alpha, g(\alpha))$ for all $\alpha \in U$, which determines its restriction to the dense subset $\pi^{-1}(U)$ of $\widetilde{X}$ and shows (by separatedness) that there is at most such map. We define κ as follows: Write $g_j = \sum_i a_{ji} f_i$ and $f_k = \sum_j b_{kj} g_j$ for some coefficients $a_{ij} \in A(X)$. For each $\alpha \in X$, we can combine the coefficients $a_{ji}(\alpha)$ and $b_{kj}(\alpha)$ together into a pair of linear transformations

$$A(\alpha) := (a_{ji}(\alpha))_{j,i} : \mathbb{P}^r \to \mathbb{P}^s$$

and

$$B(\alpha) := (b_{kj})_{k,j} : \mathbb{P}^s \to \mathbb{P}^r$$

so that $A(\alpha) f(\alpha) = g(\alpha)$ and $B(\alpha) g(\alpha) = f(\alpha)$, so that in particular, $B(\alpha) A(\alpha) f(\alpha) = f(\alpha)$. It follows that the identity $(\alpha, \beta) = (\alpha, B(\alpha) A(\alpha) \beta)$ holds on $\pi^{-1}(U)$, hence on its closure $\widetilde{X}$. In particular, $A(\alpha)\beta \neq 0$ whenever $(\alpha, \beta) \in \widetilde{X}$. It thus makes sense to define a continuous map $\kappa : \widetilde{X} \to X \times \mathbb{P}^s$ by the formula

$$\kappa((\alpha, [\beta])) := (\alpha, [A(\alpha)\beta]).$$

For all $\gamma = (\alpha, [\beta])$ in the dense open subset $\pi^{-1}(U)$ of X, we have $\gamma = (\alpha, [f(\alpha)])$ and hence $\kappa(\gamma) = (\alpha, [A(\alpha) f(\alpha)]) = (\alpha, g(\alpha))$ belongs to $\widetilde{X}'$; by continuity, it follows that $\kappa(\widetilde{X}) \subset \widetilde{X}'$. One verifies similarly that the map $\widetilde{X}' \ni (\alpha, [\beta]) \mapsto (\alpha, [B(\alpha)\beta])$ defines an inverse to κ, so that κ is an isomorphism, and that $\pi' \circ \kappa = \pi$.

TODO: probably delete.

One can also describe the blow-up $\pi : \widetilde{X} \to X$ by means of a universal property. Doing so precisely would require a bit more vocabulary than we have introduced, but the basic idea is that $\widetilde{X}$ is the smallest cover of X on which the preimage of Y is cut out locally by a single equation. TODO: see section where this is essentially verified

17.1.5 Commutative algebraic incarnation

TODO: some discussion of the blow-up ideal, perhaps via reference to Atiyah–Macdonald, seems appropriate here

17.2 The blow-up of affine space at the origin

We apply the general construction of Section 17.1.1 to affine space $\mathbb{A}^n$ together with its coordinate functions $x_1, \ldots, x_n$. The open set U of $\mathbb{A}^n$ on which some x_i doesn't vanish is given by $U = \mathbb{A}^n - \{O\}$ where $O := (0, \ldots, 0)$. The subset U is dense. By the discussion of Section 17.1.2, the corresponding blowup $\pi : \widetilde{\mathbb{A}^n} \to \mathbb{A}^n$ of $\mathbb{A}^n$ with respect to $x_1, \ldots, x_n$ is a closed subset of $\mathbb{A}^n \times \mathbb{P}^{n-1}$ on which (with the usual coordinates $x_1, \ldots, x_n$ for the $\mathbb{A}^n$ factor and $y_1, \ldots, y_n$ for the $\mathbb{P}^{n-1}$ factor) the identities

$$x_i y_j = x_j y_i$$

hold. The exceptional set is $E := \pi^{-1}(O)$. We know that π restricts to an isomorphism on the complement of E, so that in particular, the fiber above each $\alpha \in \mathbb{A}^n - O$ is the single point $\pi^{-1}(\alpha) = (\alpha, [\alpha])$. We claim that $E = \{O\} \times \mathbb{P}^{r-1}$, i.e., that the fiber of π above the origin O is the full projective space. To see this, let $[\beta] \in \mathbb{P}^{r-1}$ be given. We wish to show that $(O, [\beta]) \in \widetilde{\mathbb{A}}$. To that end, consider the corresponding line $\ell := \{\gamma\beta : \gamma \in k\}$ through the origin in $\mathbb{A}^n$. The complement in ℓ of the origin $\ell - O = \{\gamma\beta : \gamma \in k^\times\}$ has preimage

$$\pi^{-1}(\ell - O) = \{(\gamma\beta, [\beta]) : \gamma \in k^\times\} \subset \pi^{-1}(U).$$

The variety

$$Z := V_{\mathbb{A}^n \times \mathbb{P}^{n-1}}(x_i\beta_j = x_j\beta_i, y_i\beta_j = y_j\beta_i) = \{(\gamma\beta, [\beta]) : \gamma \in k\} = \ell \times \{[\beta]\}$$

contains $\pi^{-1}(\ell - O)$, and is irreducible, being isomorphic to the line ℓ. The subset $\pi^{-1}(\ell - O)$ of the irreducible variety Z is cut out by the open conditions $y_i \neq 0$, and is thus open and dense. Therefore $\overline{\pi^{-1}(\ell - O)} = Z$. In particular, $\widetilde{\mathbb{A}^n}$ contains $(O, [\beta])$, as required.

In summary, under $\pi : \widetilde{\mathbb{A}^n} \to \mathbb{A}^n$, the preimage of a nonzero point $\alpha \neq O$ is just the single point $(\alpha, [\alpha])$, while the preimage of the origin O is the exceptional set $E := \{O\} \times \mathbb{P}^{n-1}$ which identifies with the projective space of lines through the origin. We intuitively think of $\widetilde{\mathbb{A}^n}$ as the affine plane $\mathbb{A}^n$ but with the origin "blown up." The *strict transform*

$$\widetilde{X} = \overline{\pi^{-1}(X - O)} \subset \widetilde{\mathbb{A}^n}$$

of any variety $X \subset \mathbb{A}^n$ has intersection

$$\widetilde{X} \cap E$$

with the exceptional set E corresponding to the "slopes of the curves in X as they pass through the origin O." For the sake of illustration: given two curves C_1, C_2 in $\mathbb{A}^n$ containing O, their strict transforms $\widetilde{C_1}, \widetilde{C_2}$ have nonempty overlap in the exceptional set E if and only if C_1, C_2 "have a slope in common" near O. This intuition should gradually become clear as we work out examples and draw pictures.

We have focused here and shall focus henceforth on blowups at tuples $f_1, \ldots, f_r$ that cut out a point (rather than, say, some positive-dimensional variety). Similar discussions apply more generally.

17.3 The blow-up of a point on an affine variety

In the previous section, we saw that the blow-up $\widetilde{\mathbb{A}^n}$ of affine n-space at the origin O (i.e., with respect to the standard coordinate functions that define it) is the subset of $\mathbb{A}^n \times \mathbb{P}^{n-1}$ (with the affine variables $x_1, \ldots, x_n$ and projective variables $y_1, \ldots, y_n$) cut out by the equations $x_i y_j = x_j y_i$ for all pairs of indices $i, j \in \{1..n\}$. It comes equipped with a projection morphism

$$\pi_{\mathbb{A}^n,O} : \widetilde{\mathbb{A}^n} \to \mathbb{A}$$

(which one sometimes also calls the blow-up) given by $\pi_{\mathbb{A}^n,O}(\alpha, \beta) := \alpha$ for $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{A}^n$ and $\beta = [\beta_1, \ldots, \beta_n] \in \mathbb{P}^{n-1}$. We saw for a point $\alpha \neq O$ that its preimage under the blow-up is the singleton $\pi^{-1}_{\mathbb{A}^n,O}(\alpha) = \{(\alpha, [\alpha])\}$ consisting of the pair of the point α together with the unique element $[\alpha] \in \mathbb{P}^{n-1}$ containing O and α; here we identify as usual projective space $\mathbb{P}^{n-1}$ with the set of lines in $\mathbb{A}^n$ containing O. On the other hand, $\pi^{-1}_{\mathbb{A}^n,O}(O) = \{O\} \times \mathbb{P}^{n-1}$. Thus $\widetilde{\mathbb{A}^n}$ is like $\mathbb{A}^n$, but with a copy of projective space $\mathbb{P}^{n-1}$ replacing the origin (and glued together in a particular way). The restriction of $\pi_{\mathbb{A}^n,O}$ to $\widetilde{\mathbb{A}^n} - \pi^{-1}_{\mathbb{A}^n,O}(O)$ is an isomorphism onto its image $\mathbb{A}^n - O$. It is thus often convenient to identify $\mathbb{A}^n - O$ with its preimage under $\pi_{\mathbb{A}^n,O}$.

More generally, given an affine variety $X \subset \mathbb{A}^n$ and a point $\alpha \in X$, one can define its blow-up $\widetilde{X}$ at α by applying the general construction of Section 17.1.1 to the standard generators of the corresponding maximal ideal $(x_1 - \alpha_1, \ldots, x_n - \alpha_n)$. Alternatively (thanks to Section 17.1.3), $\widetilde{X}$ may be computed by first translating X so that α lies at the origin O of $\mathbb{A}^n$ and then by computing the closure

$$\widetilde{X} = \widetilde{\mathbb{A}^n} \cap (X - O)$$

of the remainder of X inside the blow-up of affine space as considered in the previous section. Thus to compute the blow-up $\widetilde{X}$ of X at (without loss of generality) the origin $O \in X \subset \mathbb{A}^n$, one should:

- Write down some defining equations

  $$g_1 = 0,$$
  $$\ldots,$$
  $$g_k = 0$$

  for X, with $g_1, \ldots, g_k \in k[x_1, \ldots, x_n]$, where $x_1, \ldots, x_n$ are coordinates on $\mathbb{A}^n$;

- Write down the equations

  $$y_i x_j = y_j x_i$$

  where the $y_1, \ldots, y_n$ are homogeneous coordinates on $\mathbb{P}^{n-1}$;

- Work on each affine open $D(y_i)$, where we dehomogenize by setting $y_i = 1$ and leave the other y_j's alone. Work on the complement $\pi^{-1}(X-O)$ of the exceptional subset of $\widetilde{X}$; equivalently, work inside the subset of $D(y_i) \cap \widetilde{X}$ cut out by the single inequation

$$x_i \neq 0,$$

noting that otherwise the identities

$$x_j = y_j x_i$$

(which follow from the earlier equations together with the dehomogenization $y_i := 1$) would force each x_j to be zero. Simplify the system of equations under consideration on this affine patch (using that x_i is invertible), typically by extracting unnecessary factors of x_i.

- Check that the simplified set of equations defines a closed subset Y of $\mathbb{A}^n$ in which $\pi^{-1}(X - O)$ is dense. For this, it suffices in to verify that Y is irreducible, i.e., that its vanishing ideal (cut out by the simplified system of equations or perhaps the radical thereof) is prime. It follows that $Y = \widetilde{X}$.

We shall work out several examples that hopefully elucidate the basic strategy here. The final step can be made a bit more systematic by a variant of the Grobner basis approach discussed above in Section 11.11, but when X is a hypersurface, it's simple enough to work directly as we shall see below. (We refrain from a completely thorough treatment; see Gathmann's notes for something more in that direction.)

In summary, for a variety $X \subset \mathbb{A}^n$ containing O, the blow-up $\widetilde{X}$ at O can be described as the closure $\widetilde{X}$ of $\pi_{\mathbb{A}^n,O}^{-1}(X - O)$ inside $\widetilde{\mathbb{A}^n}$. It comes equipped with a projection morphism $\pi_{X,O} : \widetilde{X} \to X$ given by the restriction $\pi_{X,O} := \pi_{\mathbb{A}^n,O}|_{\widetilde{X}}$ of the blow-up of affine space. The blow-up at other points of X than the origin may be defined by translating.

17.4 The blow-up of a point on any variety

Let X be a variety and $\alpha \in X$ a point. By Remark , one can (up to isomorphism) define the blow-up of X at α by first choosing passing to an affine chart $\alpha \in U \subset X$ with $\iota : U \hookrightarrow \mathbb{A}^n$ taking α to the origin O, and then glueing.

17.5 Example: the nodal cubic

Consider the variety

$$X := V(x_2^2 = x_1^2(x_1 + 1)) \subset \mathbb{A}^2.$$

As seen above, its blow-up $\widetilde{X}$ is cut out by the equations

$$x_2^2 = x_1^2(x_1 + 1)$$

$$x_1 y_2 = x_2 y_1.$$

- Work first on the affine open $y_1 = 1$ and away from the exceptional set, so that x_1 is invertible. Then the second equation $x_2 = y_2 x_1$ transforms the first equation to
$$y_2^2 x_1^2 = x_1^2(x_1 + 1)$$
which simplifies (thanks to invertibility of x_1) to
$$y_2^2 = x_1 + 1$$
which rehomogenizes to
$$y_2^2 = y_1^2(x_1 + 1).$$

- Work next on the affine open $y_2 = 1$ and away from the exceptional set, so that from $x_1 = y_1 x_2$ we obtain
$$x_2^2 = (y_1 x_2)^2 (x_1 + 1) = y_1^2 x_2^2 (x_1 + 1)$$
which simplifies to
$$1 = y_1^2(x_1 + 1)$$
and rehomogenizes again to
$$y_2^2 = y_1^2(x_1 + 1).$$

- Inspired by the above calculations, we now guess that
$$\widetilde{X} = Y$$
where
$$Y := V_{\mathbb{A}^2 \times \mathbb{P}^1}(y_2^2 = y_1^2(x_1 + 1), y_1 x_2 = y_2 x_1).$$
Note that
$$Y \cap E = V(x_1 = x_2 = 0, y_2^2 = y_1^2) = \{((0,0),[1,1]),((0,0),[1,-1])\}$$
corresponding to the two slopes with which X passes through the origin. We have seen already that Y contains the complement of the exceptional divisor in $\widetilde{X}$, whence
$$Y \supset \widetilde{X}.$$
For the reverse inclusion, it suffices to verify that Y is irreducible; indeed, it is defined on each affine patch as the locus of an irreducible polynomial, so we are done (see Section 12.2.3).

In fact, we have
$$Y \subset D(y_1),$$
since any point
$$(\alpha, [\beta]) := ((\alpha_1, \alpha_2), [\beta_1, \beta_2]) \in Y$$

for which $\beta_1 = 0$ also satisfies $\beta_2^2 = \beta_1^2(\alpha_1 + 1) = 0$ and whence $\beta_2 = 0$, contradicting the condition $(\beta_1, \beta_2) \neq (0, 0)$ in the definition of $\mathbb{P}^1$, and the morphism

$$Y \to \mathbb{A}^1$$

$$(\alpha, [\beta]) \mapsto \beta_2/\beta_1$$

is readily seen to be an isomorphism with inverse

$$\mathbb{A}^1 \to Y$$

$$\gamma \mapsto ((\gamma^2 - 1, \gamma(\gamma^2 - 1)), [1, \gamma])$$

the latter of which we divined by massaging the defining equations for Y.

Exercise 33. Take $X := V(x_2^2 = x_1^2(x_1 + 1))$ as above. Show that the map

$$(X - O) \to \mathbb{P}^1$$

$$(\alpha_1, \alpha_2) \mapsto [\alpha_1, \alpha_2]$$

does not extend to a morphism $X \to \mathbb{P}^1$. [Note first that such an extension exists after lifting via $\pi : \widetilde{X} \to X$ but takes distinct values on the exceptional fiber $\widetilde{X} \cap E = \pi^{-1}(O)$ above the origin; now invoke the separatedness of $\mathbb{P}^1$ as in Section 13.]

17.6 Example: the standard parabola

Let's compute the blow-up $\widetilde{X}$ at the origin O of

$$X := V(x_2 = x_1^2) \subset \mathbb{A}^2.$$

- On the affine patch $y_1 = 1$ and away from the exceptional set (so that $1/x_1$ is invertible) we get from $x_2 = y_2 x_1$ that

  $$x_1^2 = y_2 x_1$$

  which simplifies to

  $$x_1 = y_2$$

  and rehomogenizes to

  $$y_1 x_1 = y_2.$$

- On the affine patch $y_2 = 1$ and away from the exceptional set we get from $x_1 = y_1 x_2$ that

  $$x_2 = y_1^2 x_2^2$$

  which simplifies to

  $$1 = y_1^2 x_2$$

  and rehomogenizes to

  $$y_2^2 = y_1^2 x_2$$

- We now guess that $\widetilde{X} = Y$ where

$$Y := V(x_1 y_2 = x_2 y_1, y_2 = y_1 x_1, y_2^2 = y_1^2 x_2).$$

Indeed, Y is irreducible, being given on each affine patch by the vanishing of a single irreducible polynomial, so we are done. Note also that

$$Y \cap E = V(x_1 = x_2 = y_2 = 0) = \{(O, [1, 0])\},$$

corresponding to the unique tangent line to X at the origin. Note also that π is an isomorphism with inverse

$$X \to \widetilde{X}$$

$$(\alpha_1, \alpha_2) \mapsto ((\alpha_1, \alpha_2), [1, \alpha_1]).$$

17.7 Example: the cuspidal cubic

Let's do the same thing with

$$X := V(x_2^2 = x_1^3)$$

for which on the affine patch $y_1 = 1$ one has $x_2 = y_2 x_1$ and hence

$$y_2^2 x_1^2 = x_1^3$$

which simplifies to the irreducible equation[19]

$$y_2^2 = x_1$$

and rehomogenizes to

$$y_2^2 = x_1 y_1^2.$$

On the other affine patch $y_2 = 1$ one has $x_1 = y_1 x_2$ and so

$$x_2^2 = y_1^3 x_2^3$$

which simplifies to the irreducible equation

$$1 = y_1^3 x_2$$

and rehomogenizes to

$$y_2^3 = y_1^3 x_2.$$

Thus, as in the previous examples, we obtain

$$\widetilde{X} = V(x_1 y_2 = x_2 y_1, y_2^2 = y_1^2 x_1, y_2^3 = y_1^3 x_2)$$

[19]By an "irreducible equation" we mean an equation defined by the vanishing of an irreducible polynomial.

for which

$$\widetilde{X} \cap E = V(x_1 = x_2 = y_2 = 0) = \{(O, [1, 0])\}$$

corresponding to the unique horizontal tangent line to X at the origin. Note also that the map

$$\widetilde{X} \to \mathbb{A}^1$$

$$(\alpha, [\beta]) \mapsto \beta_2/\beta_1$$

is an isomorphism with inverse

$$\mathbb{A}^1 \to \widetilde{X}$$

$$\gamma \mapsto ((\gamma^2, \gamma^3), [1, \gamma])$$

under which the blow-up morphism $\pi : \widetilde{X} \to X$ identifies with the map (as seen on earlier homeworks)

$$\mathbb{A}^1 \to X$$

$$\gamma \mapsto (\gamma^2, \gamma^3)$$

which is a bijective homeomorphism that is not an isomorphism.

18 Differential notions

18.1 Tangent cones, tangent spaces, smoothness

We begin by verifying some basic properties of the exceptional set $\pi^{-1}(O)$ of the blow-up $\pi : \widetilde{X} \to X$ at the origin $O := (0, \dots, 0) \in \mathbb{A}^n$ of an affine variety $X \subset \mathbb{A}^n$ containing O. (By translating, we obtain a similar analysis at other points of X. By glueing and the discussion of Section 17.4, our discussion applies to varieties that are not necessarily affine.)

Recall that the preimage $\pi^{-1}(O) \subset \widetilde{X}$ of the origin is a closed subset of the projective space $\pi_{\mathbb{A}^n,O}^{-1}(O) = \{O\} \times \mathbb{P}^{n-1} \cong \mathbb{P}^{n-1}$ and from ??? that for any closed subset Z of $\mathbb{P}^{n-1}$, we may form the *affine cone* $C_Z \subset \mathbb{A}^n$; we shall think of the latter affine space as having been defined using the affine coordinates $y_1, \dots, y_n$. In particular, we form the affine cone $C_{\pi^{-1}(O)} \subset \mathbb{A}^n$.

Definition 83. Let $X \subset \mathbb{A}^n$ be an affine variety that contains the origin O. The *tangent cone* of X at O, denoted $C_0(X) \subset \mathbb{A}^n$, is defined to be $C_O(X) := C_{\pi^{-1}(O)}$. That is to say, the tangent cone is the affine cone over the exceptional set of the blow-up of X at the given point. More generally, the tangent cone $C_\alpha(X)$ of any variety X at any point $\alpha \in X$ by choosing an affine neighborhood of α, with coordinates chosen so that α is at the origin; the definition then turns out to be well-defined (see ???).

Remark 84. The tangent cone of X at α quantifies local behavior near α in the same way that the asymptotic part (i.e., the complement of X in its projective closure) quantifies behavior near ∞.

Henceforth write $k[x] := k[x_1, \ldots, x_n]$.

Definition 85. For $f \in k[x]$ with $f(O) = 0$, write $f = f_1 + f_2 + \cdots$, where f_n for $n \geq 1$ denotes the degree n homogeneous part of f. The *linear part* of f is defined to be the component $f^{\text{linear}} := f_1$. It may be identically zero. If $f \neq 0$, then the *minimal part* $f^{\min}$ of f is defined to be the component $f^{\min} := f_r$, where r is chosen as small as possible so that $f_r \neq 0$; if $f = 0$, then we set $0^{\min} := 0$. (Recall from our discussion of projective closures that we introduced the *top part* $f^{\top}$ of any $f \in k[x]$; the definition was the same as that of $f^{\min}$ except that we took r as large as possible rather than as small as possible.)

Observe that whenever the linear term f^{linear} is nonzero, one has $f^{\text{linear}} = f^{\min}$.

Example 86. In the examples mentioned above from the previous lecture, we have

1. - $f = x_2^2 - x_1^3 - x_1^2$,
   - $f^{\min} = x_2^2 - x_1^2$,
   - $f^{\text{linear}} = 0$,
2. - $f = x_2 - x_1^2$,
   - $f^{\min} = x_2$,
   - $f^{\text{linear}} = x_2$,
3. - $f = x_2^2 - x_1^3$,
   - $f^{\min} = x_2^2$,
   - $f^{\text{linear}} = 0$.

Lemma 87. *For $X \subset \mathbb{A}^n$ an affine variety containing the origin O, the exceptional set $\pi^{-1}(O)$ of the blow-up $\pi : \widetilde{X} \to X$ is given by*

$$\pi^{-1}(O) = V_{\mathbb{A}^n \times \mathbb{P}^{n-1}}(x_1 = \cdots = x_n = 0; f^{\min}(y_1, \ldots, y_n) = 0 \text{ for all } f \in I_{\mathbb{A}^n}(X)).$$

In other words, the tangent cone is given by

$$C_O(X) = V_{\mathbb{A}^n}(f^{\min} : f \in I_{\mathbb{A}^n}(X)).$$

Moreover, if $X = V_{\mathbb{A}^n}(f)$ for some $f \in k[x]$, then $C_0(X) = V_{\mathbb{A}^n}(f^{\min})$.

Proof. The proof is similar to that of the similar assertion concerning projective closures upon replacing maxima with minima. □

Remark 88. Similar to what happened for projective closures, one can ahve $X = V(f_1, \ldots, f_r)$ and yet $C_O(X) \neq V(f_1^{\min}, \ldots, f_r^{\min})$ for $r \geq 2$. One gets equality if one uses a Groebner basis for a suitable ordering.

Now we want to talk a bit about dimension. For any variety $Z \subset X$,[20] we define the *codimension of Z with respect to X*, denoted $\operatorname{codim}_X Z$, to be

$$\operatorname{codim}_X Z := \sup\{n : \text{ there is a chain } Z = Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_n \subset X \text{ with each } Z_i \text{ closed, irreducible}\}.$$

Recalling the bijection verified in ???, one can also write

$$\operatorname{codim}_X Z = \dim \mathcal{O}_{X,Z},$$

where $\mathcal{O}_{X,Z}$ is the local ring of $\mathcal{O}_X$ at Z (see ???). For example, given a point $\alpha \in X$, the codimension of the (zero-dimensional) singleton set $\{\alpha\}$ with respect to X is

$$\operatorname{codim}_X\{\alpha\} := \dim \mathcal{O}_{X,\alpha}$$

which coincides with the maximum

$$\max_i \dim X_i$$

of the dimensions of the irreducible components X_i of X that meet the point α. In words, we shall refer to $\operatorname{codim}_X\{\alpha\}$ as the *local dimension of X at the point α*. If X is irreducible and $\alpha \in X$, then we can write simply $\operatorname{codim}_X\{\alpha\} = \dim X$.

A picture to keep in mind is when $X \subset \mathbb{A}^3 := \operatorname{Specm} k[x, y, z]$ is the variety in 3-space given as the union of the vertical line L cut out by $x = y = 0$ together with the horizontal plane P cut out by $z = 0$. One then has $\operatorname{codim}_X\{\alpha\} = 2$ for $\alpha \in P$ and $\operatorname{codim}_X\{\alpha\} = 1$ for $\alpha \in L - (L \cap P)$.

Theorem 89 (Krull's principal ideal theorem, or Hauptidealsatz)**.** *Let X be an irreducible affine variety and let $0 \neq f \in A(X)$ be a nonzero regular function on X. Then each irreducible component Z of the vanishing locus $V_X(f)$ has codimension one in X, i.e.,* $\operatorname{codim}_X Z = 1$.

We deduce this from the following algebraic fact that we defer to Atiyah–Macdonald, Chapter 10 (I think; maybe 11):

Theorem 90 (Krull's Hauptidealsatz, algebraic formulation)**.** *Let $(A, \mathfrak{m})$ be a noetherian local ring and $f \in \mathfrak{m}$ a non-zerodivisor. Then* $\dim A/(f) = \dim A - 1$.

Let us now explain how Theorem 90 implies Theorem 89. Let $0 \notin f \in A(X)$ be a nonzero regular function, and $Z \subset V_X(f)$ an irreducible component. Recall that the local ring $A := \mathcal{O}_{X,Z}$ is a local noetherian integral domain obtained as the localization $A = A(X)_{I_X(Z)}$ of the affine coordinate ring $A(X)$ of X at the prime vanishing ideal $I_X(Z)$ of Z. The maximal ideal $\mathfrak{m}$ of A is generated by the image of the prime ideal $\mathfrak{p} = I_X(Z)$ under the localization map. Because prime ideals of A correspond to prime ideals $\mathfrak{p} \subset I_X(Z)$ and hence to irreducible varieties $Y \supset Z$ inside X, one has

$$\operatorname{codim}_X Z = \dim A.$$

[20] Note that we adopt the convention of writing $A \subset B$ to denote that A is contained in B and writing $A \subsetneq B$ to denote that A is properly contained in B.

Since $f \neq 0$ and $A(X) \hookrightarrow A$ is injective ($A(X)$ being an integral domain, as X is irreducible), we know that f is a non-zerodivisor. Since f vanishes on Z, we know that f belongs to $\mathfrak{m}$. Theorem 90 (its hypothesis having just been verified) implies now that

$$\dim A = \dim A/(f) + 1.$$

Primes $\mathfrak{p} \subset A/(f)$ correspond to primes $(f) \subset \mathfrak{p} \subset A$ and hence to irreducible varieties $Y \supset Z$ inside $V_X(f)$; since Z is, by hypothesis, an irreducible component of $V_X(f)$, it follows that $A/(f)$ contains a unique prime ideal, i.e., that

$$\dim A/(f) = 0.$$

Combining the above identities, we conclude that $\operatorname{codim}_X Z = 1$. This completes the proof that Theorem 90 implies Theorem 89.

The next result asserts that the irreducible components of the tangent cone of a variety at a point all have the same dimension, equal to the local dimension of the variety at that point.

Corollary 91. *Let X be a variety, and $\alpha \in X$. Then every irreducible component of the tangent cone $C_\alpha(X)$ has dimension* $\operatorname{codim}_X\{\alpha\}$.

Sketch of proof. The question is affine-local, so we may reduce to the case that $X \subset \mathbb{A}^n$ is affine and $\alpha = O$ is the origin. Consider the affine patch $D(y_i)$ of $\mathbb{A}^n \times \mathbb{P}^{n-1}$ with coordinates $x_1, \ldots, x_n, y_1, \ldots, y_n$ as above. We may set $y_i = 1$. The identity $x_j = y_i x_j = x_i y_j$ then shows that $x_i = 0$ implies $x_j = 0$ on the patch $D(y_i)$. Thus the exceptional set $\pi^{-1}(O)$ is cut out inside $D(y_i)$ by the single equation $x_i = 0$. By Krull's theorem, each irreducible component of $\pi^{-1}(O)$ has codimension one inside $\widetilde{X}$. Therefore each irreducible component of the affine cone $C_O(X)$ has the same dimension as the local dimension $\operatorname{codim}_X\{O\}$ of X at O.

Okay, you don't understand how Krull's theorem is applied here. You know already that $\pi^{-1}(O) \cap D(y_i)$ is cut out inside $\widetilde{X} \cap D(y_i)$ by the single equation $x_i = 0$. But you don't seem to know anything about irreducibility of $\widetilde{X}$, right? TODO. It seems you have to pass first to irreducible components and verify that exceptional sets of blow-ups in the dimension ≥ 1 case are nonempty. That seems all rather elaborate. Probably not worth writing up for now. Also, you're not sure where you really need any of this stuff. $\square$

Definition 92. The *tangent space* $T_O(X)$ of $X \subset \mathbb{A}^n$ at the origin $O \in X$ is defined (for the sake of computational concreteness) to be $T_O X := V_{\mathbb{A}^n}(f^{\text{linear}} : f \in I_{\mathbb{A}^n}(X))$. The tangent space at a general point α of a general variety X is defined by taking local affine coordinates as in the definition of the tangent cone.

The tangent space is always a *linear* variety, i.e., a vector space with respect to the origin. Since $f^{\text{linear}} \neq 0$ implies $f^{\text{linear}} = f^{\min}$, we know right away that

$$\{f : f^{\text{linear}} \neq 0\} \subset \{f : f^{\min} \neq 0\}$$

and hence that

$$C_\alpha(X) \subset T_\alpha(X),$$

i.e., that the tangent cone is contained in the tangent space. In general, the tangent space may be strictly larger than the tangent cone. Let us see what happens in the examples mentioned at the beginning of the lecture.

1. In the first example (the nodal cubic), we find that the tangent cone $C_O(X)$ is the union of the lines $x_2 = \pm x_1$, while the tangent space $T_O(X)$ is the plane $\mathbb{A}^2$.

2. For the parabola, the tangent cone and the tangent space are both given by the horizontal line $x_2 = 0$.

3. For the cuspidal cubic, the tangent cone $C_O(X)$ is the horizontal line $x_2 = 0$ while the tangent space $T_O(X)$ is the plane $\mathbb{A}^2$.

Definition 93. A variety X is *non-singular* (or *smooth*, *regular*, etc.) at a point $\alpha \in X$ if the tangent cone is equal to the tangent space, i.e., if $C_\alpha(X) = T_\alpha(X)$. One otherwise says that X is *singular* (or *nonsmooth*, etc.) at α.

A disadvantage of this definition is that it is not obviously intrinsic, i.e., it might appear at first glance to depend upon the choices of affine patch involved in the various definitions. The definition is in fact intrinsic. Let us explain why. Note first that since $T_\alpha(X)$ is a linear space, it is irreducible, hence any proper subvariety has strictly smaller dimension. Therefore $C_\alpha(X) = T_\alpha(X)$ if and only if $\dim T_\alpha(X) = \dim C_\alpha(X)$. On the other hand, we saw above that $\dim C_\alpha(X) = \operatorname{codim}_X\{\alpha\}$. Therefore X is nonsingular at α if and only if

$$\dim T_\alpha(X) = \operatorname{codim}_X\{\alpha\},$$

i.e., if and only if the tangent space of X at α has the same dimension as the local dimension of X at α. In general, one only has the weaker inequality

$$\dim T_\alpha(X) \geq \operatorname{codim}_X\{\alpha\}.$$

In lecture, there followed some informal discussion of describing tangent spaces $T_\alpha(X)$ using points of X valued in the dual numbers $k[\varepsilon]/(\varepsilon^2)$. We obtained in particular:

Lemma 94. *For a variety X and $\alpha \in X$, there is a natural bijection*

$$T_\alpha(X) \xrightarrow{\cong} \operatorname{Hom}(\mathfrak{m}/\mathfrak{m}^2, k)$$

where $\mathfrak{m}$ denotes the maximal ideal of the local ring $\mathcal{O}_{X,\alpha}$.

In particular, $\dim \mathfrak{m}/\mathfrak{m}^2 = \dim T_\alpha(X)$. Since we have already seen that $\dim C_\alpha(X) = \operatorname{codim}_X\{\alpha\} = \dim \mathcal{O}_{X,\alpha}$, we obtain:

Corollary 95. *X is smooth at α if and only if $\dim \mathfrak{m}/\mathfrak{m}^2 = \dim \mathcal{O}_{X,\alpha}$, i.e., if and only if $\mathcal{O}_{X,\alpha}$ is a regular local ring, and with notation as in the previous lemma.*

In particular, we see that the definition of smoothness is intrinsic, i.e., independent of the choice of affine embedding.

Definition 96. Let X be a variety and $Z \subset X$ an irreducible subvariety. Then X is said to be *smooth at* Z if $\mathcal{O}_{X,Z}$ is a regular local ring. (This is not in general the same as being smooth at all points of Z.)

18.2 Jacobian criterion

Recall from the previous section that given an affine variety $X \subset \mathbb{A}^n$ and a point $\alpha \in X$, which we may assume to be the origin $\alpha = O := (0, \ldots, 0)$ after translating as necessary, we defined the tangent cone $C_O(X) \subset \mathbb{A}^n$ to be the affine cone over the exceptional set of the blow-up of X at O. Computationally, we saw that

$$C_O(X) = V(\{f^{\min} : f \in I(X)\})$$

where $f^{\min}$ is the sum of the smallest degree monomials occurring in f. We also defined the tangent space $T_\alpha(X)$ by replacing $f^{\min}$ with f^{linear}, the sum (quite possibly empty) of all linear monomials occurring in f. It is evident that $C_\alpha X \subset T_\alpha X$ and that the latter is a linear space. We saw some examples. We saw that X is smooth at α if and only if any of any of the following equivalent conditions hold:

- $T_\alpha X = C_\alpha X$
- $\dim T_\alpha X = \operatorname{codim}_X\{\alpha\}$
- $\dim T_\alpha X \leq \operatorname{codim}_X\{\alpha\}$.
- $\mathcal{O}_{X,\alpha}$ is a regular local ring, i.e., $\dim_k \mathfrak{m}_\alpha/\mathfrak{m}_\alpha^2 = \dim \mathcal{O}_{X,\alpha}$.

We now state the Jacobian criterion: If $I(X) = (f_1, \ldots, f_r)$, then $\alpha \in X$ is smooth if and only if the quantity

$$J := \operatorname{rank}\{\frac{\partial f_i}{\partial x_j}\}_{i,j}$$

satisfies

$$J = n - \operatorname{codim}_X\{\alpha\}.$$

Intuitively, the quantity on the RHS measures the "number of independent normal directions to X at α," so the assertion is that X is smooth at α if and only if there are as many independent linear constraints on X imposed at α as there should be. A picture depicting the example $X = V(x_1 x_2) \subset \mathbb{A}^2$ clears things up a bit. Set $f := x_1 x_2$. We have $\partial f/\partial x_1 = x_2$ and $\partial f/\partial x_2 = x_1$, hence $J = 1$ if and only if $\alpha \neq 0$. On the other hand, $\operatorname{codim}_X\{\alpha\} = 1$ for all α. Thus 0 is the only nonsingular point of X.

For the proof, we refer to Hartshorne, Sec I.5.

18.3 Smooth implies locally irreducible

In the example at the end of the preceeding subsection, observe that X has two irreducible components and that the only nonsingular point of X is at the intersection of those components. It holds more generally that any point lying at the intersection of more than one irreducible component of a variety X is non-smooth. In other words, if X is smooth at a point α, then there exists a neighborhood $\alpha \in U \subset X$ that is irreducible. We describe the latter situation by saying that X is *locally irreducible at* α, so the general claim translates to "smooth implies locally irreducible." The verificatino of this implication reduces to the algebraic fact that any regular local noetherian ring is an integral domain (see Atiyah–Macdonald). On the other hand, in the same way that one verifies that an affine variety X is irreducible if and only if $A(X)$ is an integral domain, one finds that $\mathcal{O}_{X,\alpha}$ is an integral domain if and only if X is locally irreducible at α.

18.4 Smooth-set is open

We verify here that for any variety X, the set

$$\{\alpha \in X : X \text{ is smooth at } \alpha\},$$

called the *smooth subset* (or perhaps *nonsingular part*, etc.) of X is open. By the resutl fo the previous section, we may reduce to the case that X is irreducible. The question is local, so we may assume further that $X \subset \mathbb{A}^n$ is affine. Because X is irreducible, we have $\operatorname{codim}_X\{\alpha\} = \dim X$ for all $\alpha \in X$. The smoothness at a point α is thus equivalent to the lower bound $J \geq n - \dim X$, where J is the function on X given by the rank of the Jacobian matrix as in Section 18.2. That lower bound holds if and only if at least one $(n - \dim X)$-dimensional minor of J is nonzero. Each such minor is a regular function on X (indeed, the restriction of an element of $k[x_1, \dots, x_n]$) whose nonvanishing is thus an open condition. Therefore the smooth subset is defined by the union of open conditions, hence is open.

18.5 Smooth-set is nonempty

For any nonempty variety X, the smooth subset is nonempty. To give an idea for how this is proved, consider the case of a hypersurface $X = V(f) \subset \mathbb{A}^n$, where f is irreducible and nonzero. Assume also that $\operatorname{char} k = 0$. If X fails to be smooth at every point, then the Jacobian criterion implies that $\partial f / \partial x_i$ belongs to $I(X) = (f)$. Since $\deg \partial f / \partial x_i < \deg f$ and f is irreducible, we deduce that $\partial f / \partial x_i = 0$ for all i. Because $\operatorname{char} k = 0$, it follows that f is a constant, hence (because f is nonzero) that X is empty, as required. A slightly more involved argument applies in positive characteristic and also to more general varieties than hypersurfaces.

18.6 Examples of resolving singularities on a curve via successive blow-ups

In class, I worked through the details of Hartshorne, Exercise I.5.6.

19 Some generalities on algebraic groups

19.1 A definition of "group" that doesn't refer to elements

I discussed the idea of group objects and group actions in a category, and developed some of the basic theory of affine algebraic groups, covering roughly the equivalent of pages 1–5 of the notes "Introduction to actions of algebraic groups" by Michael Brion.

19.2 Algebraic groups / group varieties

- Mention connected component, actions

19.3 Basic examples

- finite groups, GL(n), closed subgroups, SL(n), O/U/Sp, split torus, multiplicative group, additive group; why it's called a torus

19.4 Morphisms

19.5 Affine implies linear

- Rational actions of an algebraic group on a vector space

19.6 Jordan decomposition is intrinsic

20 Some basics on toric varieties

Some references for the topics discussed in lecture were linked on the course website.

A *normal variety* is a variety X for which each local ring $\mathcal{O}_{X,\alpha}$ ($\alpha \in X$) is integrally closed.

A *toric variety* is a normal variety X with an action $X \circlearrowleft T$ by some torus $T = T_n$ such that X has a dense open T-orbit with trivial stabilizer.

Since T is irreducible, it follows that X is irreducible.

T_n is itself a toric variety with the usual action $T_n \circlearrowleft T_n$.

$\mathbb{A}^n$ is a toric variety with action given by multiplication

$$(\alpha_1, \dots, \alpha_n)(\tau_1, \dots, \tau_n) = (\alpha_1\tau_1, \dots, \alpha_n\tau_n).$$

The orbit of $x_0 = (1, \dots, 1)$ is dense and open.

$\mathbb{P}^1 \times \cdots \times \mathbb{P}^1 \circlearrowleft \mathbb{T}_n$ is a toric variety with the action

$$([\alpha_1, \beta_1], \dots, [\alpha_n, \beta_n])(\tau_1, \dots, \tau_n) = ([\tau_1\alpha_1, \beta_1], \dots, [\tau_n\alpha_n, \beta_n]).$$

Thus we can have many toric varieties associated to the same torus.

I briefly mentioned at the end of the lecture that toric varieties are classified by "fans" (without yet defining the latter) and drew some pictures illustrating the case $n = 2$. The following couple of lectures and homework developed the toric variety/fan correspondence in a bit more detail (see references on the course homepage).

21 Images of morphisms

I started with the motivating problem: given some polynomial equations or inequations in the variables $\alpha_1, \dots, \alpha_m, \beta_1, \dots, \beta_n$, in which $\beta = (\beta_1, \dots, \beta_n)$ is regarded as a parameter and $\alpha = (\alpha_1, \dots, \alpha_m)$ as the unknown, can one say something about the set of β for which there exists a solution α? Some basic examples involving determinants, resultants and discriminants were discussed. The problem was then reformulated in more general terms as describing images of (certain classes of) morphisms between varieties. The basic example $(x, y) \mapsto (xy, y)$ was discussed. Chevalley's theorem that the image of a constructible set is constructible was stated and briefly applied, without proof.

Here we shall record the key definitions and results were as follows. (These were discussed with somewhat more motivation and in somewhat more piecemeal steps in lecture.) Let X be a variety. A subset $C \subset X$ is called *constructible* if it is a boolean combination of subvarieties. In other words, the collection of constructible subsets of X is the smallset such collection that contains every open or closed subset and that is preserved under taking complements and finite unions and finite intersections.

Theorem 97. *Let $f : X \to Y$ be a morphism of varieties. Then the image $f(X)$ is constructible. In fact, $f(C)$ is constructible for all constructible $C \subset X$.*

We did not prove this result but indicated the idea briefly.

Next, say that a variety X is *proper* if for each variety Z, the projection map $\pi : X \times Z \to Z$ given by $(\alpha, \beta) \mapsto \beta$ is closed, i.e., $\pi(E)$ is closed for all closed $E \subset X \times Z$. (This is similar to the definition of a "compact" topological space.)

Theorem 98. *Every projective variety is proper.*

This result has many consequences:

Corollary 99. *Let $f : X \to Y$ be a morphism of varieties, with X projective. Then $f(X)$ is closed. More generally, $f(E)$ is closed for all closed subvarieties E of X.*

Proof. We know (see ???) that the graph $\Gamma_f \subset X \times Y$ is closed, so by Theorem 98, it follows that $\pi(\Gamma_f)$ is closed. It is clear that $f(X) = \pi(\Gamma_f)$, where $\pi : X \times Y \to Y$ is the natural projection onto the second factor. Therefore $f(X)$ is closed. The second assertion follows similarly. $\square$

Recall that every irreducible variety is connected.

Corollary 100. *Let X be a connected projective variety. Then every regular function on X is constant, i.e., $\mathcal{O}_X(X) = k$.*

Proof. Let $f \in \mathcal{O}_X(X)$. Regard f as a morphism $f : X \to \mathbb{A}^1$. Denote by $f' : X \to \mathbb{P}^1$ the composition of f with the inclusion $\mathbb{A}^1 \hookrightarrow \mathbb{P}^1$. Then $f'(X)$ is a closed and connected subset of $\mathbb{P}^1$ that is not equal to all of $\mathbb{P}^1$. Any such subset consists of a single point, so f' (and hence f) is constant. $\square$

Example 101. $\mathbb{A}^1$ is not proper because the projection map $\pi : \mathbb{A}^1 \times \mathbb{A}^1 \to \mathbb{A}^1$ given by $(\alpha, \beta) \mapsto \beta$ is not closed: for instance, the hyperbola $V(xy = 1) \subset \mathbb{A}^1 \times \mathbb{A}^1$ is closed, but maps under π to the complement $\mathbb{A}^1 - \{0\}$ of the origin, which is not closed.

Remark 102. Analogues of the above results hold for compact complex manifolds.

For the proof of Theorem 98, we reduced (by some elementary arguments) to showing that the projection map $\pi : \mathbb{P}^m \times \mathbb{P}^n \to \mathbb{P}^m$ is closed. To establish the latter fact, take $Z = V(f_1, \dots, f_r)$ with $f_1, \dots, f_r \in k[x_0, \dots, x_m, y_0, \dots, y_n]$ bihomogeneous of some bidegree (d, d), say (see §15.5). The aim is to show that $\pi(Z)^c$ is open. Each of the following conditions on a point $\alpha \in \mathbb{P}^m$ is equivalent to the next:

- α belongs to $\pi(Z)^c$.
- Write $g_i := f_i(\alpha, \cdot) \in k[y_0, \dots, y_n]$; it is homogeneous of degree d. There does not exist β so that $g_1(\beta) = \cdots = g_r(\beta) = 0$.
- There exists $N \geq 1$ so that $(g_1, \dots, g_r) \subset S_N$, where $S := k[y_0, \dots, y_n]$ and S_N denotes the Nth graded component. (We have used the projective Nullstellensatz.)
- The map $\mu : S_{N-d} \times \cdots \times S_{N-d} \to S_N$ given by $(h_1, \dots, h_r) \mapsto h_1 g_1 + \cdots + h_r g_r$ is surjective. (Indeed, the image of this map is the Nth graded component of the ideal $(g_1, ceds, g_r)$.) Note that μ is k-linear, and described by a finite-dimensional matrix M whose entrise are homogeneous polynomials in the coefficients of the $g_1, \dots, g_r$, hence homogeneous polynomials in α.
- Some $(\dim S_N) \times (\dim S_N)$-dimensional minor of M is nonzero.

The final condition, being a union of open conditions on α, is open. Thus $\pi(Z)^c$ is open.

22 Classification of curves up to birational equivalence

I more-or-less covered the results of Hartshorne, I.6, but gave somewhat different proofs. A good reference is Brian Osserman's note "Nonsingular curves."

We defined (at first) a *curve* to be an irreducible variety of dimension one. Note that any nonempty open subset of a curve is a curve. The set of non-smooth points on a curve is finite. Thus, every curve contains a smooth curve as an open subset. In particular, every curve is birational to a nonsingular curve. The principal result of this section is that in fact, every curve is birational to a *unique* nonsingular *projective* curve. Moreover, morphisms between nonsingular projective curves are in natural arrow-reversing bijection with morphisms between their function fields, which are precisely the field extensions of transcendence degree one over k.

A key notion is that of a *uniformizer* at a smooth point $\alpha \in C$ on a curve C. It follows from some commutative algebra that $\alpha \in C$ is smooth if and only if any of the following equivalent criteria hold:

1. The local ring $\mathcal{O}_{C,\alpha}$ is a regular local ring (of dimension one).
2. $\mathcal{O}_{C,\alpha}$ is integrally closed.
3. $\mathcal{O}_{C,\alpha}$ is a discrete valuation ring, i.e., there exists a surjective "order of vanishing" map $v_\alpha : \mathcal{O}_{C,\alpha} \to \mathbb{Z}_{\geq 0} \cup \{\infty\}$ satisfying some properties. We may extend it to $v_\alpha : k(C) \to \mathbb{Z} \cup \{$
$infty\}$. Then $\mathcal{O}_{C,\alpha} = \{f \in k(C) : v_\alpha(f) \geq 0\}$.
4. The maximal ideal $\mathfrak{m}_\alpha$ of $\mathcal{O}_{C,\alpha}$ is principal.

A *uniformizer* t for C at α is a rational function $t \in k(C)$ which is regular at α and whose image in $\mathcal{O}_{C,\alpha}$ generates the maximal ideal $\mathfrak{m}_\alpha$. The normalization is such that $v_\alpha(t) = 1$.

We can think of a uniformizer as an equivalence class of pairs (U, t) where $\alpha \in U \subset C$ is a neighborhood and $t \in \mathcal{O}_C(U)$ is a regular function for which $v_\alpha(t) = 1$. For example, $t := x - \alpha$ is a uniformizer at a point $\alpha \in \mathbb{A}^1$. In general, we may think of t as a regular function defined in a neighborhood of α that has a simple zero at α. We may assume, after shrinking U sufficiently, that t has no zeros in U other than α. We then have that for each smaller neighborhood $\alpha \in V \subset U$:

1. for $f \in \mathcal{O}_C(V)$, we have $f(\alpha) = 0$ if and only if $t|_V$ divides f inside $\mathcal{O}_C(V)$;
2. for a nonzero $f \in \mathcal{O}_C(V - \{\alpha\})$, there exists $g \in \mathcal{O}_C(V)$ so that $f = t^\nu g$ on their common domain, with $\nu := v_\alpha(f)$. We have $\nu \geq 0$ iff f extends to a regular function on all of V. If $\nu < 0$, we say that f has (at α) a *pole* of order $-\nu$. For example, $1/t^3$ has a pole of order 3 at α.

We discussed the following four examples of morphisms $f : X - \{P\} \to Y$ from a variety minus a point to another variety:

Example 103.

1. Take $X := \mathbb{A}^1$, $p := 0$, $Y := \mathbb{A}^2$, and $f(x) := (x, 1/x)$. Then $f : \mathbb{A}^1 - \{0\} \to \mathbb{A}^2$ does not extend to a morphism $\mathbb{A}^1 \to \mathbb{A}^2$. (We've seen this before.)

2. Take $X := \mathbb{A}^2, p := 0$, $Y := \mathbb{P}^1$, and $f(\alpha_1, \alpha_2) := [\alpha_1, \alpha_2]$. Then $f : \mathbb{A}^2 - \{0\} \to \mathbb{P}^1$ does not extend to a morphism $\mathbb{A}^2 \to \mathbb{P}^1$ (We've seen this before.)

3. Take $X := \mathbb{A}^1, p := 0, Y := \mathbb{P}^2$, and $f(x) := [x, 1/x, 1]$. Then $f : \mathbb{A}^1 - \{0\} \to \mathbb{P}^1$ extends to a morphism $f : \mathbb{A}^1 \to \mathbb{P}^2$, given explicitly by $f(x) := [x^2, 1, x]$.

4. Take $X := V_{\mathbb{A}^2}(y^2 - x^2(x-1))$, $p := 0$, $Y := \mathbb{P}^1$, and $f(x, y) := [x, y]$. Then $f : X - \{0\} \to \mathbb{P}^1$ does not extend to a morphism $f : X \to \mathbb{P}^1$. (We saw this in some previous lecture.)

The above examples demonstrate the necessity of the hypotheses in the following result:

Theorem 104. *Let C be a curve, Y a projective variety, $p \in C$ a smooth point, and $f : C - \{p\} \to Y$ a morphism. Then there is a unique extension of f to a morphism $C \to Y$.*

Proof. Uniqueness follows as usual from the separatedness of Y, using that any closed subset of C that contains $C - \{p\}$ is equal to C itself. For existence, say $Y \subset \mathbb{P}^n$. Represent f in some puctured neighborhood $V - \{p\}$ of p as $f = [f_0, \dots, f_n]$ with each $f_i \in \mathcal{O}(V - \{p\}) \subset k(C)$. After shrinking V as necessary, take a uniformizer $t \in \mathcal{O}(V)$ for the point p which vanishes at no other point of V. Set $\nu_0 := v_p(f_0), \dots, \nu_n := v_p(f_n)$, and write $f_i = t^{\nu_i} g_i$ with $g_i \in \mathcal{O}(V)$ and $g_i(p) \neq 0$ unless $f_i \equiv 0$; leave out any $f_i \equiv 0$ in the argument to follow. Set $k := \min\{\nu_0, \dots, \nu_n\} \in \mathbb{Z}$. Then $f = [t^{-k} f_0, \dots, t^{-k} f_n]$. Each $t^{-k} f_i$ extends to a regular function at P because $v_p(t^{-k} f_i) = \nu_i - k \geq 0$. Some $t^{-k} f_i$ is nonzero at p because for some i, we have $k = \nu_i$; in that case, $t^{-k} f_i(p) = 0$. (The proof is thus the same as in the exercise that any rational map $\mathbb{P}^1 \to \mathbb{P}^n$ extends to a morphism, and also essentially the same as in the first of the examples above.) $\square$

Corollary 105. *If $p_1, \dots, p_n$ are smooth points on a curve C and Y is a projective variety, then any morphism $C - \{p_1, \dots, p_n\} \to Y$ extends uniquely to $C \to Y$.*

Corollary 106. *If C is a curve, $U \subset C$ is a nonempty open, $C - U$ is nonsingular (inside C), and Y is projective, then any $U \to Y$ extends uniquely to $C \to Y$.*

Corollary 107. *If C_1, C_2 are nonsingular projective curves that are birational, then $C_1 \cong C_2$. In other words, open subsets of a nonsingular projective curve determine the isomorphism class of the curve. (Proof given in class.)*

Varieties need not be quasi-projective; patching affines together doesn't necessarily keep one in the same projective space. However:

Theorem 108. *Let C be a nonsingular curve. Then C is quasi-projective: there exists a projective curve $\overline{C} \subset \mathbb{P}^n$ and an isomorphism from C to an open subset of $\overline{C}$. (Proof given in class, using patching on affines, Segre embedding, and Theorem 104.)*

In class, we defined the normalization of a curve and showed that the normalization of any projective curve is a nonsingular curve. (The normalization of a variety X is a normal variety $\widetilde{X}$ and a dominant morphism $\widetilde{X} \to X$ such that for all normal varieties Y and dominant morhpisms $Y \to X$, there exists a unique $Y \to \widetilde{X}$. The construction of X is given by "take integral closures on each affine patch and glue.")

We then explained how, given a curve X, one can take a nonsingular open $U \subset X$, realize it as a quasi-projective curve, take its projective closure C, and then take the normalization $\widetilde{C}$ to end up with a nonsingular projective curve that is birational to the original curve (see the references from the beginning of this section for further details).

Corollary 109. *For each nonsingular curve C there exists a nonsingular projective curve $\overline{C}$ such that C is isomorphic to an open subvariety of $\overline{C}$. We call $\overline{C}$ "the" nonsingular compactification of C.*

Finally, we recorded that:

Corollary 110. *Let $\varphi : C_1 \to C_2$ be a non-constant morphism of curves, with C_1 projective. Then if φ is surjective.*

Proof. C_1 is proper, so $\varphi(C_1)$ is closed. C_1 is connected, so $\varphi(C_1)$ is connected. The only closed connected subsets of C_2 are C_2 itself and singleton sets consisting of points. Since φ is non-constant, it follows that $\varphi(C_1) = C_2$. $\square$

Corollary 111. *Let $\varphi : C_1 \to C_2$ be a non-constant morphism between curves (not necessarily projective or nonsingular). Then $\varphi(C_1)$ is open. (This is a special case of Chevalley's theorem.)*

Proof. The question is local, so we may assume that C_1, C_2 are nonsingular. We extend φ to a map $\overline{\varphi} : \overline{C_1} \to \overline{C_2}$ between nonsingular compactifications. By the previous corollary, we know that $\overline{\varphi}$ is surjective. Therefore $\varphi(C_1)$ has finite complement, as required. $\square$

23 Nonsingular projective curves

A reference is Chapter 2 of Silverman's "The Arithmetic of Elliptic Curves." There is also Chapter II.6 in Hartshorne, but the language there is somewhat different from what was discussed in lecture. I also see that Brian Ossmerman's notes "Divisors on nonsingular curves" and "Differential forms" are relevant.

Henceforth "curve" means "nonsingular projective curve." Thus uniformizers exist at all points. From the previous section, we know that morphisms between curves are the same as rational maps, or as morphisms between their function fields in the opposite direction.

Given $f : X \to Y$ a non-constant (hence surjective, and in particular, dominant) morphism of curves, we get an induced map of function fields $f^\sharp : k(Y) \hookrightarrow k(X)$ via pullback. Identify $k(Y)$ with its image in $k(X)$. The *degree of* f is $\deg(f) := [k(X) : k(Y)]$. It is finite.

23.1 Review of uniformizers

23.2 Degree of a morphism

23.3 Order of vanishing of a regular function

23.4 Ramification indices of a morphism

23.5 Examples

23.6 Sum formula for degree of a morphism

- Chinese remainder theorem formula for degree of a morphism as a sum over preimages of a given point

23.7 Divisors

23.8 Rational functions have divisors of degree zero

23.9 Picard group

23.10 Linear systems

23.10.1 Definition

23.10.2 Basic properties

23.10.3 Connection with effective linear divisors equivalent to a given one

23.11 Analogues in Riemann surface theory

23.12 Divisor short exact sequence

23.12.1 Main section

23.12.2 Analogue over number fields

23.13 The projective line has trivial Picard group

23.14 Differentials on a curve

23.15 Riemann–Roch

23.16 The group law on elliptic curves

24 Etc

In the final week, I gave a quick overview of the theory of schemes and sketched a proof of the Hasse bound for elliptic curves.