SELECTED SOLUTIONS FOR ALGEBRAIC GEOMETRY SPRING SEMESTER 2015

Sheet 1

Exercise 2

a) $SL_2(\mathbb{C})$ is equal to the variety in $\mathbb{A}^4$ cut out by the equation $detA - 1 = 0$, where A is a matrix in 4 indeterminates.

b) We want to show that any polynomial that vanishes at every element of $SL_2(\mathbb{Z})$ must also vanish at every element of $SL_2(\mathbb{C})$. Let $P(x_{11}, x_{12}, x_{21}, x_{22})$ be such a polynomial. Let $\gamma \in SL_2(\mathbb{C})$. Then, as stated in the exercise,

$$\gamma = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

can be written as a product of the form

$$\gamma = \begin{pmatrix} 1 & b_1 \\ c_1 & 1 \end{pmatrix} \begin{pmatrix} 1 & b_2 \\ c_2 & 1 \end{pmatrix}_{...} \begin{pmatrix} 1 & b_k \\ c_k & 1 \end{pmatrix}$$

where $b_i, c_i \in \mathbb{C}$ and for each $i = 1, ..., k$ either c_i or b_i is zero.

WLOG, assume that

$$\gamma = \begin{pmatrix} 1 & \beta_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \beta_2 & 1 \end{pmatrix}_{...} \begin{pmatrix} 1 & 0 \\ \beta_k & 1 \end{pmatrix}$$

Thus the a_{ij} are polynomials in the β_l, i.e. we have poylnomials $\alpha_{ij}(y_1, ..., y_k)$ such that $a_{ij} = \alpha_{ij}(\beta_1, ..., \beta_k)$.

Then $P(\alpha_{11}, ..., \alpha_{22})$ is a polynomial Q in the y_j and, since the matrix M given by $(M)_{ij} = \alpha_{ij}(z)$ is in $SL_2(\mathbb{Z})$ for all $z \in \mathbb{Z}^k$, P vanishes at M and so Q vanishes on $\mathbb{Z}^k$. By the same argument as $1d$), such a polynomial must vanish on $\mathbb{C}^k$. In particular, $Q(\beta_1, ..., \beta_k) = P(a_{11}, ..., a_{22}) = 0$. ■

Exercise 5

Let $\Gamma := \{a_1, ..., a_m\} \subset \mathbb{A}^n$ be a finite set of points.

Writing $a_i = (a_{i1}, ..., a_{in})$ for $i = 1, ..., m$, we can assume WLOG that $a_{ij} \neq a_{kj}$ for $k \neq i$. Indeed, there are infinitely many lines through a single point in Γ (since k is infinite) and only finitely many whose translations to the other points in Γ contain more than one point in Γ. Choose a line l outside of this finite set. Repeating the process, noting that there are infinitely many lines not contained in a given linear subspace of $\mathbb{A}^n$ of dimension strictly smaller than n, we can attain n linearly-independent such lines and use them to parameterize $\mathbb{A}^n$.

Now define

$$L_k(x_1, ..., x_{n-1}) := \frac{\Pi_{i=1, i\neq k}^m (x_1 - a_{i1})}{\Pi_{i=1, i\neq k}^m (a_{k1} - a_{i1})} ... \frac{\Pi_{i=1, i\neq k}^m (x_{n-1} - a_{i(n-1)})}{\Pi_{i=1, i\neq k}^m (a_{k(n-1)} - a_{i(n-1)})}$$

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Then, $L_k(a_j) = \delta_k^j$ and writing x for $(x_1, ..., x_{n-1})$ and defining

$$p(x) := a_{1n}L_1(x) + ... + a_{mn}L_k(x)$$

we see that $p(a_{i1}, ..., a_{in-1}) = a_{in}$ for $i = 1, ..., m$. This implies that the variety $Y \subset \mathbb{A}^n$ defined by

$$Q(x_1, ..., x_n) := p(x) - x_n$$

contains Γ.

We proceed by induction. In the case $n = 1$, we simply take the product of $(x - a_i)$ where a_i runs over all the points in Γ.

Now suppose $n > 1$ and that any finite set of points in $\mathbb{A}^{n-1}$ can be written as the zero locus of $n-1$ polynomials. Consider the projection map $\pi : \mathbb{A}^n \to \mathbb{A}^{n-1}$ sending $(y_1, ..., y_n) \to (y_1, ..., y_{n-1})$. By assumption there exist polynomials $q_1, ..., q_{n-1}$ in $x_1, ..., x_{n-1}$ such that $V_{\mathbb{A}^{n-1}}(q_1, ..., q_{n-1}) = \pi(\Gamma)$.

Then $V := \pi^{-1}(\pi(\Gamma)) = V_{\mathbb{A}^n}(q_1, ..., q_{n-1})$ consists precisely of the lines of the form $l_i = (a_{i1}, ..., a_{i(n-1)}, X)$, with $i = 1, ..., m$ and X runs through k. Now, with Q as above, $\Gamma \subset V(Q)$ and Q cannot vanish anywhere else on the lines l_i since, writing $a_i = (a_{i1}, ..., a_{i(n-1)})$, if Q vanishes on (a_i, λ) and (a_i, λ'), with $\lambda \neq \lambda'$, then $p(a_i) = \lambda$ and $p(a_i) = \lambda'$, which is impossible. Therefore $V \cap V(Q) = V(q_1, ..., q_{n-1}, Q) = \Gamma$.■

Sheet 4

Exercise 4

Let Γ be a finite subset of $\mathbb{P}^n$ in general position of cardinality $|\Gamma| = d \leq 2n$.

Recall that a finite set of points in $\mathbb{P}^n$ is in general position if the liftings of any m points to lines in $\mathbb{A}^{n+1}$, where $0 \leq m \leq n+1$, via the quotient map $\pi : \mathbb{A}^n - \{0\} \to \mathbb{P}^n$ generates a linear subspace of dimension m. We say that the points span an $m-1$-dimensional linear subspace of $\mathbb{P}^n$.

See *Harris, Algebraic Geometry, Theorem 1.4* for the proof in the case $d = 2n$. We will assume in the following that this case is proven and proceed by induction.

Suppose $1 \leq d < 2n$ and that any collection of $d+1$ points in general position can be described as the zero locus of quadratic polynomials.

We claim that there exists a point $p' \in \mathbb{P}^n$ not contained in Γ such that the set $\Gamma' = \Gamma \cup \{p'\}$ is in general position. Indeed, if $d < n+1$ we simply choose any point not contained in the linear subspace of $\mathbb{P}^n$ generated by Γ. Otherwise, we must find a point p' such that the linear subspace generated by p' and any n points in Γ has dimension n. Since only $\begin{pmatrix} d \\ n \end{pmatrix}$ hyperplanes can be spanned by n points in Γ, and there are infinitely many points not contained in the union of these hyperplanes, such a p' must exist.

By assumption the set $\Gamma' = V(Q_1, ..., Q_k)$ where the Q_i are quadratic polynomials. Choose $S_1, S_2 \subset \Gamma$ with $|S_i| \leq n$ and $\Gamma = S_1 \cup S_2$. Then the linear subspaces Λ_i spanned by the S_i are the zero locii of linear forms and since Γ' is in general position, $p' \notin \Lambda_1 \cup \Lambda_2$. Since $\Gamma \subset \Lambda_1 \cup \Lambda_2$, which is also described by quadratics $P_1, ..., P_j$, we have $\Gamma = \Gamma' \cap (\Lambda_1 \cup \Lambda_2) = V(Q_1, .., Q_k, P_1, ..., P_j)$. ■